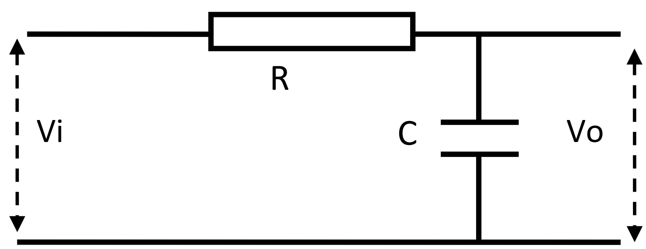
LECTURE ONE

Mathematical Models of Systems

1.1 Transfer Function:

The T.F. of a System is the ratio of Laplace Transforms of the output and input quantities, where initial conditions being Zero.

Example 1.1: Derive the T.F. (Vo/Vi) of the below shown circuit.



Solution:

Using voltage divider rule;

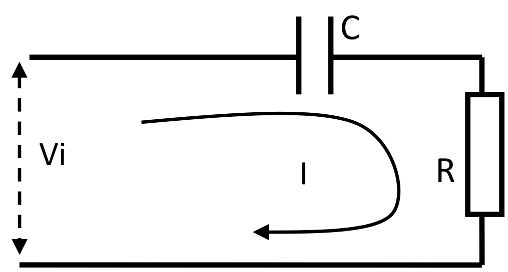
$$V_o = V_i \frac{X_C}{X_C + R}$$

Note that $X_C = \frac{1}{j\,2\,\pi\,f\,C} = \frac{1}{S\,C}$

Therefore; $\frac{V_O}{V_i} = \frac{\frac{1}{S\,C}}{R+\frac{1}{S\,C}}$, Simplifying to obtain; $\frac{V_O}{V_i} = \frac{1}{SCR+1} = \frac{1}{ST+1}$

Where, T=RC.

Example 1.2: Derive the T.F. (I/Vi) of the below shown circuit.



Solution:

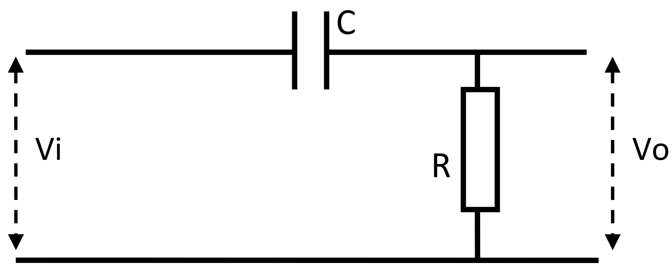
$$V_o = Z_{total}\ I = I\ (R + \frac{1}{SC})$$

Then;

$$\frac{I}{V_o} = \frac{1}{R + \frac{1}{SC}} = \frac{SC}{SCR + 1}$$

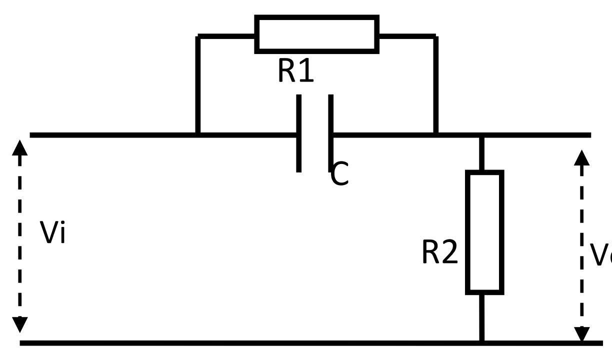
Problem 1.1:

Derive the transfer function (V_o/V_i) for the circuit shown below;



<u>**Example 1. 3:**</u>

Derive the transfer function (V_o/V_i) for the circuit shown below.



Solution:

$$V_o = V_i \frac{R2}{Z_1 + R2}$$

$$\frac{V_o}{V_i} = \frac{R2}{Z_1 + R2}$$

Where $Z_1 = R1//X_C$

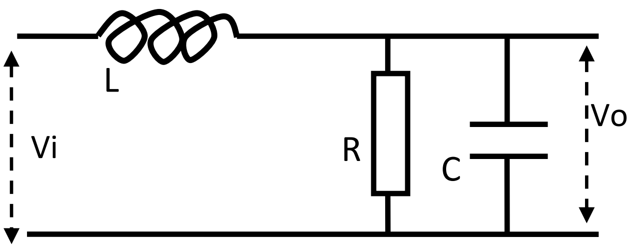
$$Z_1 = \frac{R1\, X_C}{R1 + X_C} = \frac{R1\, \frac{1}{SC}}{R1 + \frac{1}{SC}} = \frac{R1}{R1CS + 1}$$

Then;

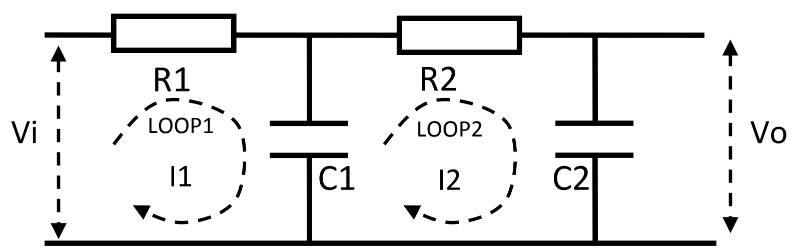
$$\frac{V_o}{V_i} = \frac{R2}{\frac{R1}{R1CS+1} + R2} = \frac{R1\, R2\, C\, S + R2}{R1 + R1R2CS + R2}$$

Problem 1.2:

Derive the transfer function (V_o/V_i) for the circuit shown below;



Example 1.4: Derive the transfer function (V_o/V_i) for the circuit shown below;



Solution:

Constructing KVL equations.

LOOP1:

$Vi = I1R1 + X_{C1}\ (I1-I2)$

$Vi = (X_{C1}+R1)\ I1 - X_{C1}\ I2$ ……………………… (1.1)

LOOP2:

$X_{C1}\ (I2-I1) + R2\ I2 + Vo = 0$

$(X_{C1}+R2)\ I2 - X_{C1}\ I1 + Vo = 0$ …………………………. (1.2)

Now since V0= I2 X_{C2} then substituting in equation (1) will result in:

Vi= (XC1+R1) I1- X_{C1} (V0/ X_{C2})

$$I1 = \frac{Vi + \frac{X_{C1}}{X_{C2}} V0}{X_{C1} + R1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.3)$$

then substituting equation (3) together with V0= I2 X_{C2} in equation (2), get;

$$(X_{C1} + R2)\frac{Vo}{X_{C2}} - X_{C1}\frac{Vi + \frac{X_{C1}}{X_{C2}} V0}{X_{C1} + R1} + Vo = 0$$

$$\frac{(X_{C1} + R2)}{X_{C2}} Vo - \frac{{X_{C1}}^2}{X_{C2}X_{C1} + R1X_{C2}} V0 + V0 - \frac{X_{C1}}{X_{C1} + R1} Vi = 0$$

$$\frac{Vo}{Vi} = \frac{\frac{X_{C1}}{X_{C1} + R1}}{\frac{(X_{C1} + R2)}{X_{C2}} - \frac{{X_{C1}}^2}{X_{C2}X_{C1} + R1X_{C2}} + 1}$$

$$\frac{Vo}{Vi} = \frac{X_{C1}}{\frac{(X_{C1} + R1)(X_{C1} + R2)}{X_{C2}} - \frac{{X_{C1}}^2}{X_{C2}} + (X_{C1} + R1)}$$

$$\frac{Vo}{Vi} = \frac{X_{C1}\, X_{C2}}{(X_{C1} + R1)(X_{C1} + R2) - {X_{C1}}^2 + X_{C2}R1 + X_{C2}X_{C1}}$$

$$\frac{Vo}{Vi} = \frac{X_{C1}\, X_{C2}}{{X_{C1}}^2 + R1R2 + R1\, X_{C1} + R2\, X_{C1} - {X_{C1}}^2 + X_{C2}R1 + X_{C2}X_{C1}}$$

$$\mathbf{\frac{Vo}{Vi} = \frac{X_{C1}\, X_{C2}}{R1R2 + R2X_{C1} + R1X_{C1} + X_{C2}R1 + X_{C2}X_{C1}}}$$

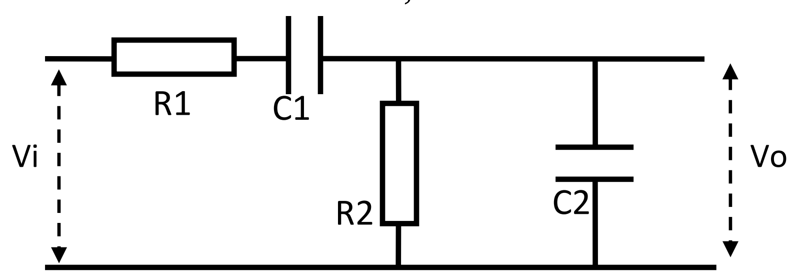
Therefore;

$$\frac{\mathbf{Vo(s)}}{\mathbf{Vi(s)}} = \frac{\frac{1}{SC_1}\frac{1}{SC_2}}{\mathbf{R1R2} + \mathbf{R2}\frac{1}{SC_1} + \mathbf{R1}\frac{1}{SC_1} + \frac{1}{SC_2}\mathbf{R1} + \frac{1}{SC_1}\frac{1}{SC_2}}$$

$$\frac{\mathbf{Vo(s)}}{\mathbf{Vi(s)}} = \frac{1}{C_1C_2\mathbf{R1R2}\,S^2 + \mathbf{R2}\,C_2S + \mathbf{R1}C_2S + \mathbf{R1}C_1S + 1}$$

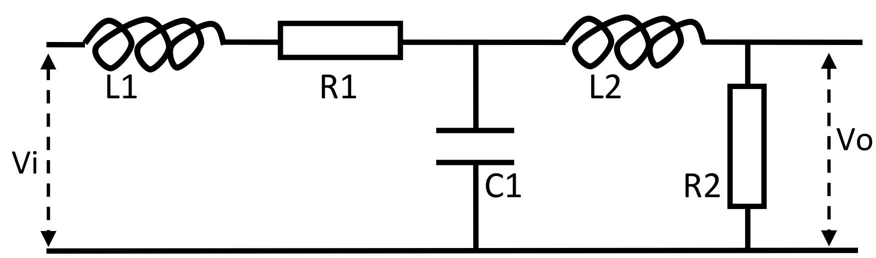
Problem 1.3:

Derive the transfer function (V_o/V_i) for the circuit shown below;



Problem 1.4:

Derive the transfer function (V_o/V_i) for the circuit shown below;



Problem 1.5:

Find the transfer function (V_o/V_i) for the circuit shown below;