



# LECTURE FOUR

## TIME DOMAIN ANALYSIS

### TRANSIENT AND STEADY-STATE RESPONSE ANALYSIS

#### 4.1 Transient Response and Steady-State Response

The time response of a control system consists of two parts: the transient response and the steady-state response. By transient response, we mean that which goes from the initial state to the final state. By steady-state response, we mean the manner in which the system output behaves as it approaches infinity. Thus the system response  $c(t)$  may be written as

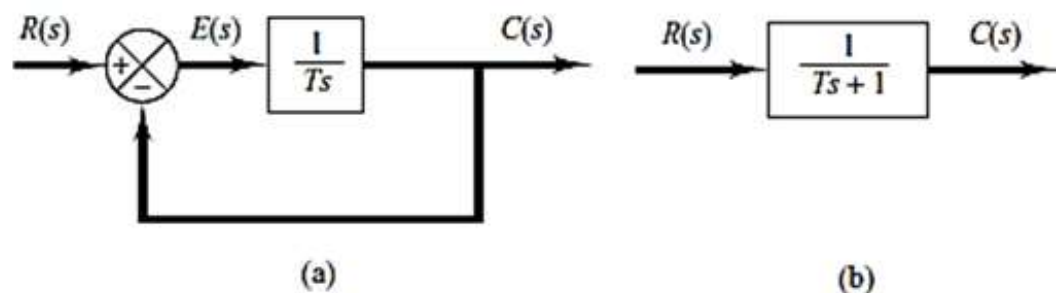
$$c(t) = c_{tr}(t) + c_{ss}(t) \quad (4.1)$$

#### 4.2 First-Order Systems

Consider the **first-order** system shown in Figure 4.1-a below. Physically, this system may represent an  $RC$  circuit, thermal system, or the like. A simplified block diagram is shown in Figure 4.1-b. The input-output relationship is given by

$$\frac{C(s)}{R(s)} = \frac{1}{Ts + 1}$$

**Figure 4.1**  
(a) Block diagram of a first-order system;  
(b) simplified block diagram.





In the following, we shall analyze the system responses to such inputs as the unit-step, unit-ramp, and unit-impulse functions. The initial conditions are assumed to be zero.

### 4.2.1 Unit-Step Response of First-Order Systems

Since the Laplace transform of the unit-step function is  $1/s$ , substituting  $R(s) = 1/s$  into Equation (4.1), we obtain

$$C(s) = \frac{1}{Ts + 1} \frac{1}{s}$$

Expanding  $C(s)$  into partial fractions gives

$$C(s) = \frac{1}{s} - \frac{T}{Ts + 1} = \frac{1}{s} - \frac{1}{s + (1/T)}$$

Taking the inverse Laplace transform, we obtain

$$c(t) = 1 - e^{-t/T}, \quad \text{for } t \geq 0 \quad (4.2)$$

Equation (4.2) states that initially the output  $c(t)$  is zero and finally it becomes unity. One important characteristic of such an exponential response curve  $c(t)$  is that at  $t=T$  the value of  $c(t)$  is 0.632, or the response  $c(t)$  has reached 63.2% of its total change. This may be easily seen by substituting  $t=T$  in  $c(t)$ . That is,

$$c(T) = 1 - e^{-1} = 0.632$$

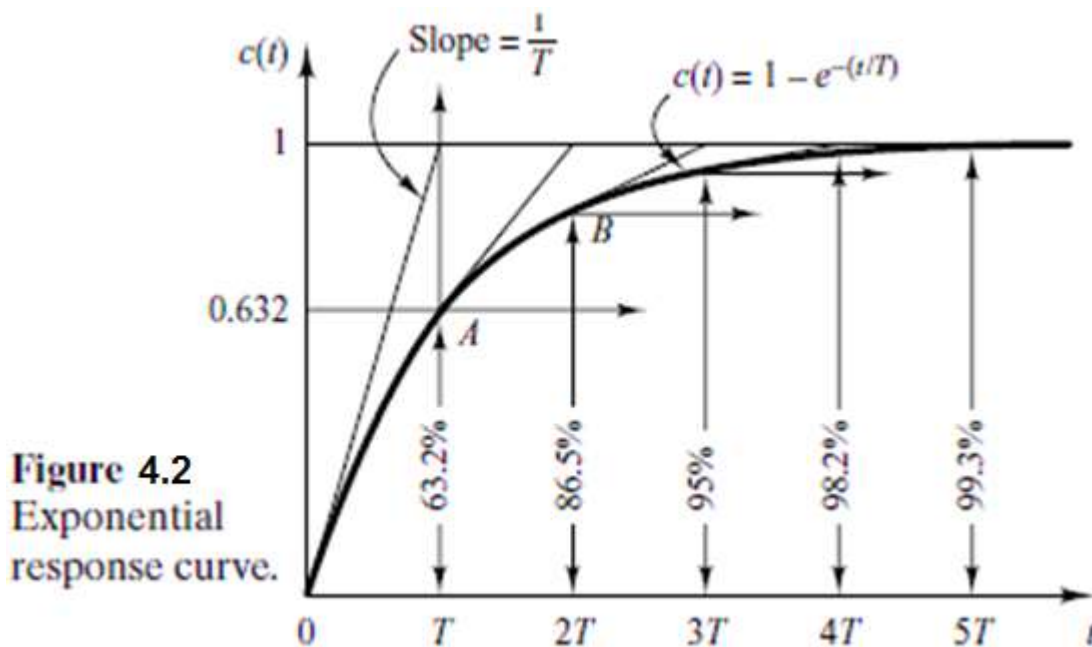
Note that the smaller the time constant  $T$ , the faster the system response. Another important characteristic of the exponential response curve is that the slope of the tangent line at  $t=0$  is  $1/T$ , since

$$\left. \frac{dc}{dt} \right|_{t=0} = \left. \frac{1}{T} e^{-t/T} \right|_{t=0} = \frac{1}{T}$$



The output would reach the final value at  $t=T$  if it maintained its initial speed of response. From Equation above we see that the slope of the response curve  $c(t)$  decreases monotonically from  $1/T$  at  $t=0$  to zero at  $t=\infty$ .

The exponential response curve  $c(t)$  given by Equation (4.2) is shown in Figure 4.2. In one time constant, the exponential response curve has gone from 0 to 63.2% of the final value. In two time constants, the response reaches 86.5% of the final value. At  $t=3T$ ,  $4T$ , and  $5T$ , the response reaches 95%, 98.2%, and 99.3%, respectively, of the final value. Thus, for  $t \geq 4T$ , the response remains within 2% of the final value. As seen from Equation (4.2), the steady state is reached mathematically only after an infinite time. In practice, however, a reasonable estimate of the response time is the length of time the response curve needs to reach and stay within the 2% line of the final value, or four time constants.





### 4.2.2 Unit-Ramp Response of First-Order Systems

Since the Laplace transform of the unit-ramp function is  $1/s^2$ , we obtain the output of the system of Figure 1(a) as

$$C(s) = \frac{1}{Ts + 1} \frac{1}{s^2}$$

Expanding  $C(s)$  into partial fractions gives

$$C(s) = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{Ts + 1}$$

Taking the inverse Laplace transform of above equation, we obtain

$$c(t) = t - T + Te^{-t/T}, \quad \text{for } t \geq 0$$

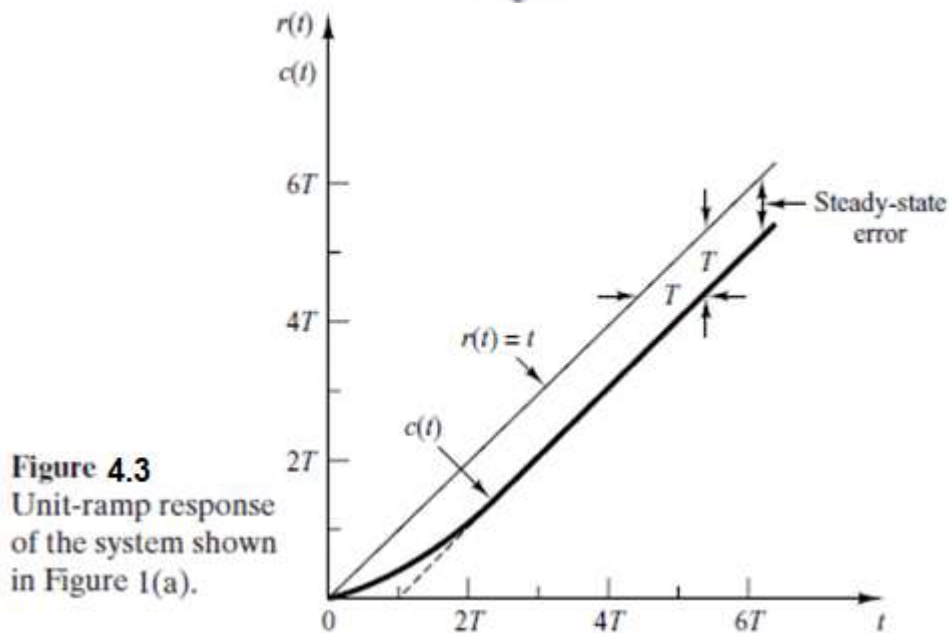
The error signal  $e(t)$  is then

$$\begin{aligned} e(t) &= r(t) - c(t) \\ &= T(1 - e^{-t/T}) \end{aligned}$$

As  $t$  approaches infinity,  $e^{-t/T}$  approaches zero, and thus the error signal  $e(t)$  approaches  $T$  or

$$C(s) = \frac{1}{Ts + 1}$$

The unit-ramp input and the system output are shown in Figure 4.3. The error in following the unit-ramp input is equal to  $T$  for sufficiently large  $t$ . The smaller the time constant  $T$ , the smaller the steady-state error in following the ramp input.



### 4.2.3 Unit-Impulse Response of First-Order Systems

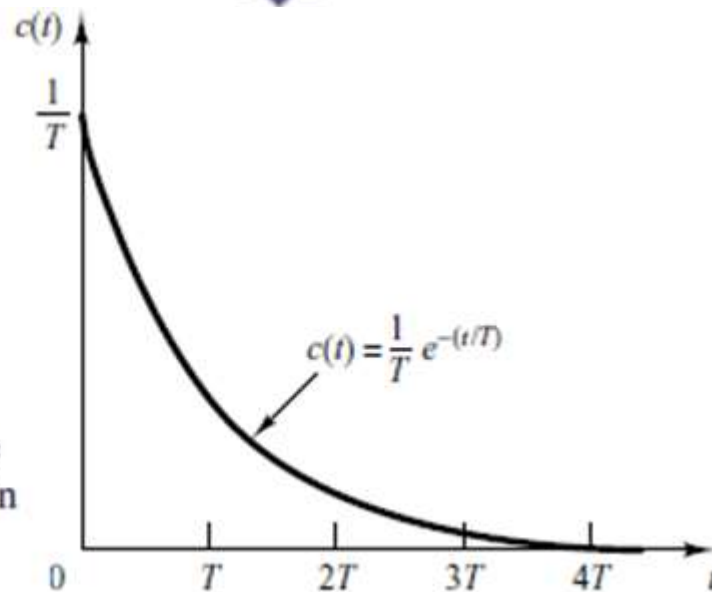
For the unit-impulse input,  $R(s)=1$  and the output of the system of Figure 4.1(a) can be obtained as

$$C(s) = \frac{1}{Ts + 1}$$

The inverse Laplace transform of above Equation

$$c(t) = \frac{1}{T} e^{-t/T}, \quad \text{for } t \geq 0 \quad (4.3)$$

The response curve given by Equation (3) is shown in Figure 4.4.



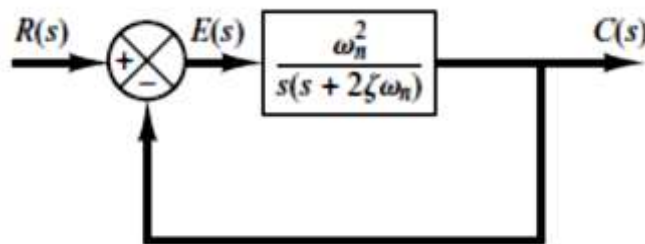
**Figure 4.4**  
Unit-impulse  
response of the  
system shown in  
Figure 1(a).

### 4.3 Second-Order Systems

Any second order system can be represented by the following typical form and system diagram:

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

(4.4)



This is called the **standard form** of the second order system, where;

$\omega_n$  = undamped natural frequency (rad/sec).

$\zeta$  (zeta) = damping ratio

**Example 4.1:**

For the system shown below, find  $\omega_n$  and  $\zeta$ .

$$\frac{C(s)}{R(s)} = \frac{k/T}{s^2 + \frac{s}{T} + \frac{k}{T}}$$

Comparing with the standard form of the 2nd order system, obtaining;

$$\omega_n^2 = \frac{k}{T} \rightarrow \omega_n = \sqrt{\frac{k}{T}} \text{ (rad/sec)}$$

also ;

$$2\zeta\omega_n = \frac{1}{T} \rightarrow \zeta = \frac{1}{2T} \sqrt{\frac{T}{k}} = \frac{1}{2\sqrt{kT}}$$

**Example 4.2:**

For the system shown below, find  $\omega_n$  and  $\zeta$ .

$$\frac{C(s)}{R(s)} = \frac{8}{2s^2 + 0.8s + 8}$$

First of all, the transfer function should be rearranged as;

$$\frac{C(s)}{R(s)} = \frac{4}{s^2 + 0.4s + 4}$$

and by comparing with the standard form, get;

$$\omega_n^2 = 4 \rightarrow \omega_n = 2 \text{ (rad/sec)}$$



also ;

$$2\zeta\omega_n = 0.4 \rightarrow \zeta = \frac{0.4}{4} = 0.1$$

We shall now solve for the response of the system shown in equation (4.4) to a **unit-step input**. We shall consider three different cases: the **underdamped** ( $0 < \zeta < 1$ ), critically damped ( $\zeta = 1$ ), and overdamped ( $\zeta > 1$ ) cases.

### **Unit Step input:**

**(1) Underdamped case** ( $0 < \zeta < 1$ ): In this case,  $C(s)/R(s)$  can be written as

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s + \zeta\omega_n + j\omega_d)(s + \zeta\omega_n - j\omega_d)}$$

Where  $\omega_d = \omega_n \sqrt{1 - \zeta^2}$ . The frequency  $\omega_d$  is called the **damped natural frequency**.

For a unit-step input,  $C(s)$  can be written

$$C(s) = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)s} \quad (4.5)$$

The inverse Laplace transform of Equation (4.5) can be obtained easily if  $C(s)$  is written in the following form:





$$\begin{aligned}
 C(s) &= \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} \\
 &= \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}
 \end{aligned}$$

Referring to the Laplace transform, it can be shown that

$$\begin{aligned}
 \mathcal{L}^{-1}\left[\frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}\right] &= e^{-\zeta\omega_n t} \cos \omega_d t \\
 \mathcal{L}^{-1}\left[\frac{\omega_d}{(s + \zeta\omega_n)^2 + \omega_d^2}\right] &= e^{-\zeta\omega_n t} \sin \omega_d t
 \end{aligned}$$

Hence the inverse Laplace transform of Equation (4.5) is obtained as

$$\begin{aligned}
 \mathcal{L}^{-1}[C(s)] &= c(t) \\
 &= 1 - e^{-\zeta\omega_n t} \left( \cos \omega_d t + \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin \omega_d t \right) \\
 &= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1 - \zeta^2}} \sin \left( \omega_d t + \tan^{-1} \frac{\sqrt{1 - \zeta^2}}{\zeta} \right), \quad \text{for } t \geq 0
 \end{aligned} \tag{4.6}$$

From Equation (4.6), it can be seen that the frequency of transient oscillation is the damped natural frequency  $\omega_d$  and thus varies with the damping ratio  $\zeta$ . The error signal for this system is the difference between the input and output and is

$$\begin{aligned}
 e(t) &= r(t) - c(t) \\
 &= e^{-\zeta\omega_n t} \left( \cos \omega_d t + \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin \omega_d t \right), \quad \text{for } t \geq 0
 \end{aligned}$$



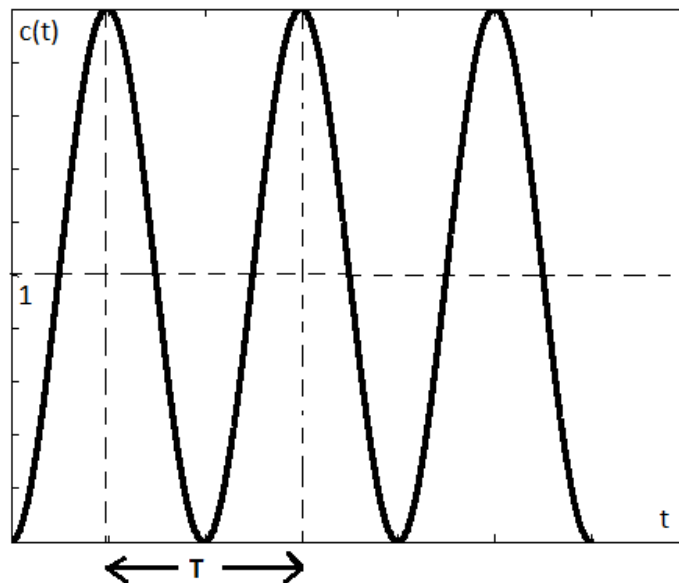
This error signal exhibits a damped sinusoidal oscillation. At steady state, or at  $t = \infty$ , no error exists between the input and output.

**(2) Undamped case** ( $\zeta=0$ ):

If the **damping ratio  $\zeta$  is equal to zero**, the response becomes **undamped and oscillations continue indefinitely**. The response  $c(t)$  for the zero damping case may be obtained by substituting  $\zeta=0$  in Equation (6), yielding

$$c(t) = 1 - \cos \omega_n t, \quad \text{for } t \geq 0 \quad (4.7)$$

Therefore, the time of oscillation ( $T$ ) is  $T = \frac{2\pi}{\omega_n}$



Thus, from Equation (4.7), we see that  $\omega_n$  represents the **undamped natural frequency** of the system. That is,  $\omega_n$  is that frequency at which the system output would oscillate if the damping were decreased to zero. If the linear system has any amount of damping, the undamped natural frequency cannot be observed experimentally. The frequency that may be observed is the damped natural frequency  $\omega_d$ , which is equal to  $\omega_n \sqrt{1 - \zeta^2}$ . This frequency



is always lower than the undamped natural frequency. An increase in  $\zeta$  would reduce the damped natural frequency  $\omega_d$ . If  $\zeta$  is increased beyond unity, the response becomes overdamped and will not oscillate.

### **(3) Critically damped case** ( $\zeta=1$ ):

If the two poles of  $C(s)/R(s)$  are equal, the system is said to be a critically damped one. For a unit-step input,  $R(s)=1/s$  and  $C(s)$  can be written

$$C(s) = \frac{\omega_n^2}{(s + \omega_n)^2 s} \quad (4.8)$$

The inverse Laplace transform of Equation (8) may be found as

$$c(t) = 1 - e^{-\omega_n t}(1 + \omega_n t), \quad \text{for } t \geq 0 \quad (4.9)$$

This result can also be obtained by letting  $\zeta$  approach unity in Equation (4.6) and by using the following limit:

$$\lim_{\zeta \rightarrow 1} \frac{\sin \omega_d t}{\sqrt{1 - \zeta^2}} = \lim_{\zeta \rightarrow 1} \frac{\sin \omega_n \sqrt{1 - \zeta^2} t}{\sqrt{1 - \zeta^2}} = \omega_n t$$

### **(4) Overdamped case** ( $\zeta > 1$ ):

In this case, the two poles of  $C(s)/R(s)$  are negative real and unequal. For a unit-step input,  $R(s)=1/s$  and  $C(s)$  can be written



$$C(s) = \frac{\omega_n^2}{(s + \zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1})(s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1})s} \quad (4.10)$$

The inverse Laplace transform of Equation (10) is

$$\begin{aligned} c(t) &= 1 + \frac{1}{2\sqrt{\zeta^2 - 1}(\zeta + \sqrt{\zeta^2 - 1})} e^{-(\zeta + \sqrt{\zeta^2 - 1})\omega_n t} \\ &\quad - \frac{1}{2\sqrt{\zeta^2 - 1}(\zeta - \sqrt{\zeta^2 - 1})} e^{-(\zeta - \sqrt{\zeta^2 - 1})\omega_n t} \\ &= 1 + \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} \left( \frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2} \right), \quad \text{for } t \geq 0 \end{aligned} \quad (4.11)$$

Where  $s_1 = (\zeta + \sqrt{\zeta^2 - 1})\omega_n$  and  $s_2 = (\zeta - \sqrt{\zeta^2 - 1})\omega_n$ .

Thus, the response  $c(t)$  includes two decaying exponential terms. When  $\zeta$  is appreciably greater than unity, one of the two decaying exponentials decreases much faster than the other, so the faster-decaying exponential term (which corresponds to a smaller time constant) may be neglected. That is, if  $-s_2$  is located very much closer to the  $j\omega$  axis than  $-s_1$  (which means  $|s_2| \gg |s_1|$ ), then for an approximate solution we may neglect  $-s_1$ . This is permissible because the effect of  $-s_1$  on the response is much smaller than that of  $-s_2$ , since the term involving  $s_1$  in Equation (4.11) decays much faster than the term involving  $s_2$ . Once the faster-decaying exponential term has disappeared, the response is similar to that of a first-order system, and  $C(s)/R(s)$  may be approximated by;



$$\frac{C(s)}{R(s)} = \frac{\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}}{s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}} = \frac{s_2}{s + s_2}$$

This approximate form is a direct consequence of the fact that the initial values and final values of the original  $C(s)/R(s)$  and the approximate one agree with each other. With the approximate transfer function  $C(s)/R(s)$ , the unit-step response can be obtained as

$$C(s) = \frac{\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}}{(s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1})s}$$

The time response  $c(t)$  is then

$$c(t) = 1 - e^{-(\zeta - \sqrt{\zeta^2 - 1})\omega_n t}, \quad \text{for } t \geq 0$$

This gives an approximate unit-step response when one of the poles of  $C(s)/R(s)$  can be neglected.

A family of unit-step response curves  $c(t)$  with various values of  $\zeta$  is shown in Figure (4.5), where the x-axis is the dimensionless variable  $\omega_n t$ . The curves are functions only of  $\zeta$ . These curves are obtained from Equations (4.6), (4.9), and (4.11). The system described by these equations was initially at rest. Note that two second-order systems having the same  $\zeta$  but different  $\omega_n$  will exhibit the same overshoot and the same oscillatory pattern. Such systems are said to have the same relative stability. From Figure (4.5), we see that an **underdamped**



system with  $\zeta$  **between 0.5 and 0.8** gets close to the final value **more rapidly** than a critically damped or over damped system. Among the systems responding without oscillation, a critically damped system exhibits the fastest response. An over damped system is always sluggish in responding to any inputs. It is important to note that, for second-order systems whose closed-loop transfer functions are different from that given by Equation (4.4), the step-response curves may look quite different from those shown in Figure 4.5.

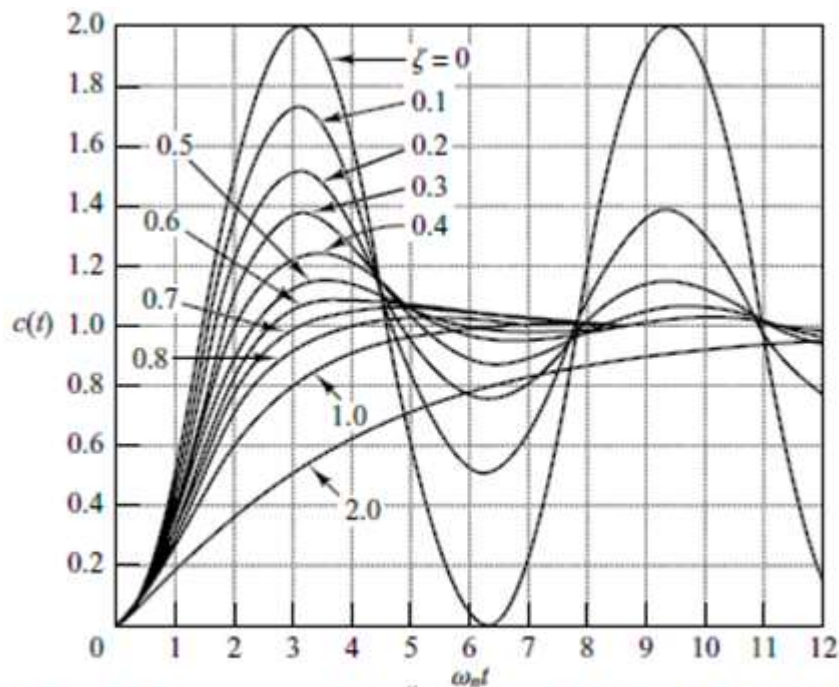


Figure 4.5 Unit-step response curves of the 2nd order system

#### 4.4 Definitions of Transient-Response Specifications

The transient response of a practical control system often exhibits damped oscillations before reaching steady state. In specifying the transient-response characteristics of a control system to a unit-step input, it is common to specify the following:



1. Delay time,  $t_d$
2. Rise time,  $t_r$
3. Peak time,  $t_p$
4. Maximum over shoot,  $M_p$
5. Settling time,  $t_s$

These specifications are defined in what follows and are shown graphically in previous Figures.

1. Delay time,  $t_d$ : The delay time is the time required for the response to reach half the final value the very first time.
2. Rise time,  $t_r$  : The rise time is the time required for the response to rise from 10% to 90%, 5% to 95%, or 0% to 100% of its final value. For underdamped second order systems, the 0%to 100%rise time is normally used. For over damped systems, the 10% to 90% rise time is commonly used.
3. Peak time,  $t_p$ : The peak time is the time required for the response to reach the first peak of the overshoot.
4. Maximum (percentage) overshoot,  $M_p$ : The maximum overshoot is the maximum peak value of the response curve measured from unity. If the final steady-state value of the response differs from unity, then it is common to use the maximum percentage overshoot. It is defined by:

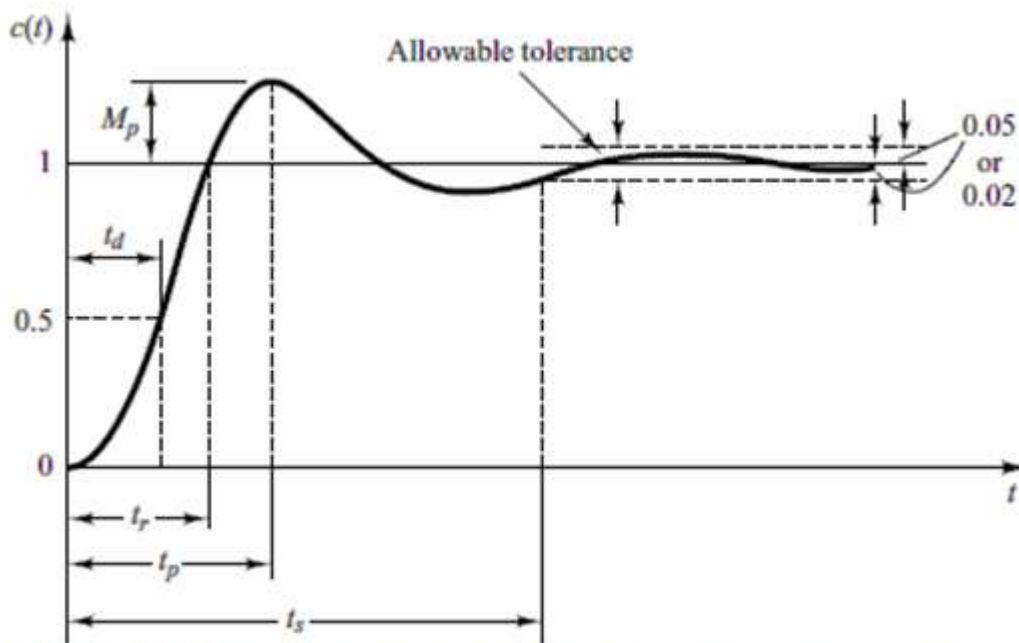
$$\text{Maximum percent overshoot} = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

The amount of the maximum (percentage) overshoot directly indicates the relative stability of the system.



**5. Settling time,  $t_s$  :** The settling time is the time required for the response curve to reach and stay within a range about the final value of size specified by absolute percentage of the final value (usually 2% or 5%). The settling time is related to the largest time constant of the control system. Which percentage error criterion to use may be determined from the objectives of the system design in question.

The time-domain specifications just given are quite important, since most control systems are time-domain systems; that is, they must exhibit acceptable time responses (This means that, the control system must be modified until the transient response is satisfactory).



**Figure 4.6** Unit-step response curve showing  $t_d$ ,  $t_r$ ,  $t_p$ ,  $M_p$ , and  $t_s$ .

It is desirable that the transient response be sufficiently fast and be sufficiently damped. Thus, for a desirable transient response of a second-order system, the damping ratio must be





between 0.4 and 0.8. Small values of  $\zeta$  (that is,  $\zeta < 0.4$ ) yield excessive overshoot in the transient response, and a system with a large value of  $\zeta$  (that is,  $\zeta > 0.8$ ) responds sluggishly.

#### 4.5 Second-Order Systems and Transient-Response Specifications:

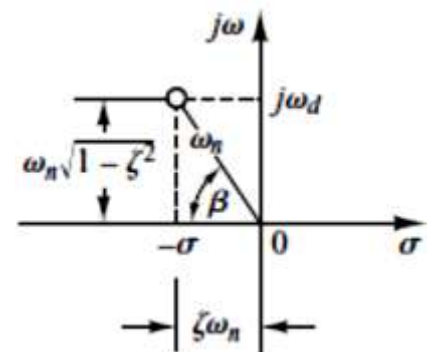
In the following, we shall obtain the **rise time**, **peak time**, **maximum overshoot**, and **settling time** of the second-order system. These values will be obtained in terms of  $\zeta$  and  $\omega_n$ . The system is assumed to be **underdamped**.

##### 1- Rise time $t_r$ :

Since  $\omega_n \sqrt{1 - \zeta^2} = \omega_d$  and  $\zeta \omega_n = \sigma$ , we have

$$t_r = \frac{\pi - \beta}{\omega_d}$$

(4,12)



Where  $\beta = \tan^{-1}\left(\frac{\omega_d}{\sigma}\right)$  ,

**2- Peak time  $t_p$ :** We may obtain the peak time by differentiating  $c(t)$  with respect to time and letting this derivative equal zero. This will result in;

$$t_p = \frac{\pi}{\omega_d}$$

(4.13)

The peak time  $t_p$  corresponds to one-half cycle of the frequency of damped oscillation.



**3- Maximum overshoot  $M_p$ :** The maximum overshoot occurs at the peak time or at  $t=t_p=\pi/\omega_d$ .  $M_p$  generally is obtained by;

$$M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \quad (4.14)$$

$$M_p = \frac{1}{c(\infty)} [c(t_p) - 1] = \frac{1}{c(\infty)} e^{-(\zeta\pi)/\sqrt{1-\zeta^2}}$$

and by assuming  $c(\infty)=1$

In general ;  $c(\infty) = \lim_{s \rightarrow 0} s c(s)$  *final value theorem*

$$M_p = e^{-(\zeta\pi)/\sqrt{1-\zeta^2}}$$

Taking the natural logarithm (ln) for both sides;

$$\ln(M_p) = -\frac{\zeta\pi}{\sqrt{1-\zeta^2}} \rightarrow \ln^2(M_p)(1-\zeta^2) = \zeta^2\pi^2$$

$$\ln^2(M_p) = \zeta^2\pi^2 + \ln^2(M_p)\zeta^2 \rightarrow \zeta^2 = \frac{\ln^2(M_p)}{\pi^2 + \ln^2(M_p)}$$

$$\zeta = \sqrt{\frac{\ln^2(M_p)}{\pi^2 + \ln^2(M_p)}}$$

**4- Settling time  $t_s$  :** The settling time corresponding to a ; 2% or ;5% tolerance band may be measured in terms of the time constant  $T=1/\zeta \omega_n$  for different values of  $\zeta$ . The results are shown in Figure (4.7). For  $0 < \zeta < 0.9$ , if the 2% criterion is used,  **$t_s$**  is approximately four times the time constant of the system. If the 5% criterion is used, then  **$t_s$**  is approximately three times the time constant. Note that the settling time reaches a minimum value around  **$\zeta=0.76$  (for the 2% criterion) or  $\zeta=0.68$  (for the 5% criterion)** and then increases almost linearly for large values of  $\zeta$ . For convenience in comparing the responses of systems, we commonly define the settling time  **$t_s$**  to be;



$$t_s = 4T = \frac{4}{\sigma} = \frac{4}{\zeta\omega_n} \quad (2\% \text{ criterion})$$

(4.15)

Or

$$t_s = 3T = \frac{3}{\sigma} = \frac{3}{\zeta\omega_n} \quad (5\% \text{ criterion})$$

(4.16)

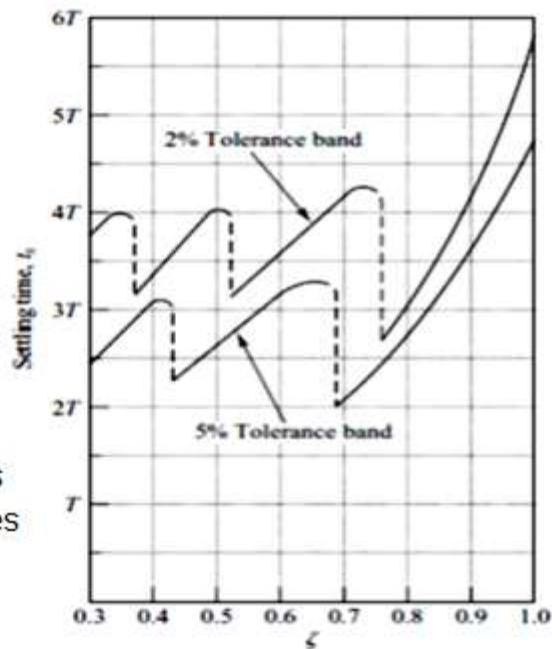


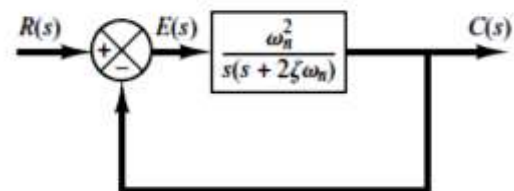
Figure 4.7  
Settling time  $t_s$   
versus  $\zeta$  curves

### Example 4.3:

Consider the system shown in figure below, where  $\zeta=0.6$  and  $\omega_n=5$  rad/sec. Find the rise time  $t_r$ , peak time  $t_p$ , maximum overshoot  $M_p$ , and settling time  $t_s$  when the system is subjected to a **unit-step** input

From the given values of  $\zeta$  and  $\omega_n$ , we obtain;

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} = 4 \text{ and } \sigma = \zeta\omega_n = 3.$$



*Rise time*  $t_r$  : The rise time is



$$t_r = \frac{\pi - \beta}{\omega_d} = \frac{3.14 - \beta}{4}$$

where  $\beta$  is given by

$$\beta = \tan^{-1} \frac{\omega_d}{\sigma} = \tan^{-1} \frac{4}{3} = 0.93 \text{ rad}$$

The rise time  $t_r$  is thus

$$t_r = \frac{3.14 - 0.93}{4} = 0.55 \text{ sec}$$

*Peak time  $t_p$* : The peak time is

$$t_p = \frac{\pi}{\omega_d} = \frac{3.14}{4} = 0.785 \text{ sec}$$

*Maximum overshoot  $M_p$* : The maximum overshoot is

$$M_p = e^{-(\sigma/\omega_d)\pi} = e^{-(3/4) \times 3.14} = 0.095$$

The maximum percentage overshoot is thus 9.5%.

$$t_s = \frac{4}{\sigma} = \frac{4}{3} = 1.33 \text{ sec}$$

For the 5% criterion,

$$t_s = \frac{3}{\sigma} = \frac{3}{3} = 1 \text{ sec}$$

#### **Example 4.4:**

Design a second order system by finding the system transfer function with response to a unit step input that ensures maximum overshoot equal or less than 10% and settling time less than 0.5 seconds. Also compute rising time, peak time, and steady state error.



Solution:

The second order standard form is;

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Therefore, it is required to find  $\zeta$  and  $\omega_n$ . Now for  $M_p=10\%=0.1$ , then

$$\zeta = \sqrt{\frac{\ln^2(M_p)}{\pi^2 + \ln^2(M_p)}} = \sqrt{\frac{\ln^2(0.1)}{\pi^2 + \ln^2(0.1)}} = 0.5911$$

And since the settling time should be less than 0.5 seconds, therefore assuming  $t_s=0.5$ , then according to the 2% criterion;

$$t_s = \frac{4}{\zeta\omega_n} \rightarrow \omega_n = \frac{4}{0.5 \cdot 0.5911} = 13.53 \text{ rad/sec}$$

Then;

$$G(s) = \frac{183}{s^2 + 15.9951 s + 183}$$

Now;

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} = 13.53 * \sqrt{1 - (0.5911)^2} = 10.91 \text{ rad/sec}$$



$$\text{and } \sigma = \zeta \omega_n = 0.5911 * 13.53 = 7.99$$

$$\text{Also } \beta = \tan^{-1} \left( \frac{\omega_d}{\sigma} \right) = \tan^{-1} \left( \frac{10.91}{7.99} \right) = 0.938$$

Therefore;

$$t_r = \frac{\pi - \beta}{\omega_d} = \frac{\pi - 0.938}{10.91} = 0.2019 \text{ seconds}$$

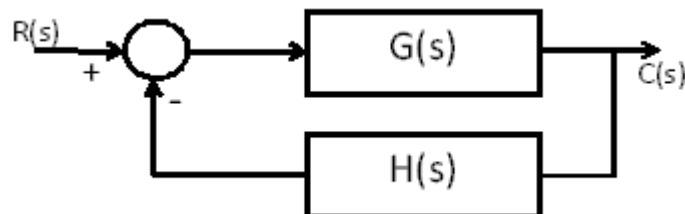
and

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{10.91} = 0.287 \text{ seconds}$$

$$c(\infty) = 1 \text{ and therefore } e_{ss} = \text{Zero}$$

## 4.6 Stability Analysis

Consider the control system block diagram below.



The transfer function of the system is



$$\frac{c(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

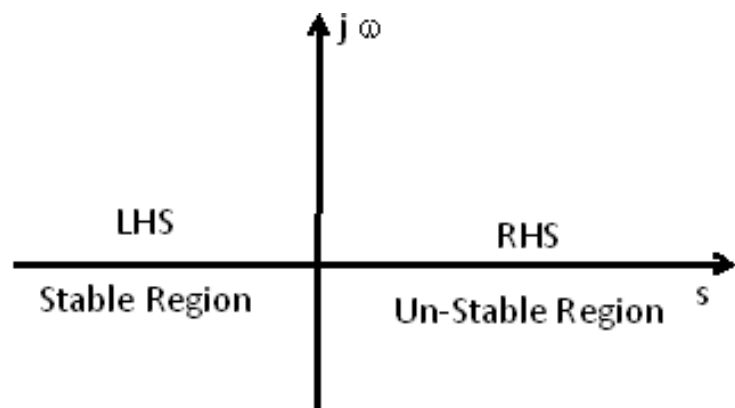
However,  $(1 + G(s)H(s))$  is called the characteristics equation of the system. Now a control system is considered stable if all the roots of the characteristics equation (C.E.) are in the Left Hand Side (LHS) of the S-plane.

**Stable Region:**

$$S = -\sigma \pm j\omega$$

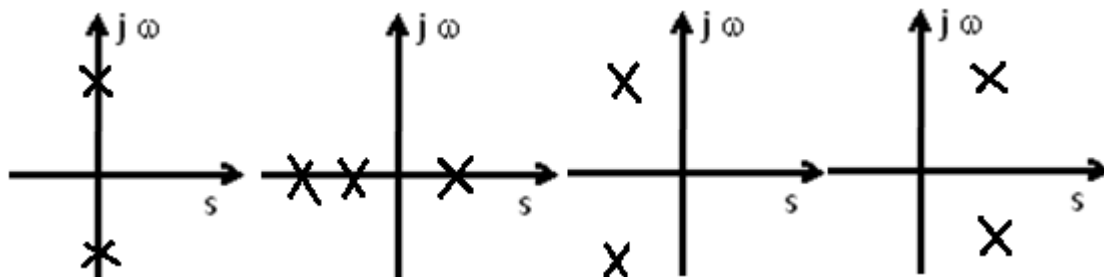
**Un-Stable Region:**

$$S = \sigma \pm j\omega$$



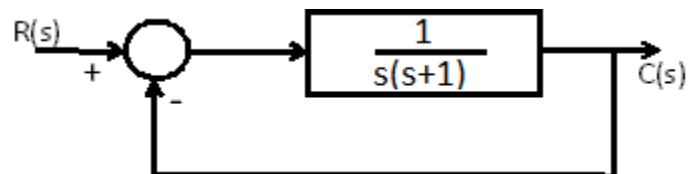
$G(s)H(s)$  is also called the open loop transfer function.

-In the following figure, some stability cases may be shown.



**Example 4.5:**

For the system shown below, state whether the system is stable or not. Also find the transient and steady state parameters.



Solution:

$$\frac{c(s)}{R(s)} = \frac{1}{s^2 + s + 1}$$

Therefore,  $s_1 = -0.5 + j$  and  $s_2 = -0.5 - j$

Since the poles  $S_1$  and  $S_2$  are at the right hand side RHS

Then the system is stable.

Comparing with the second order standard form;

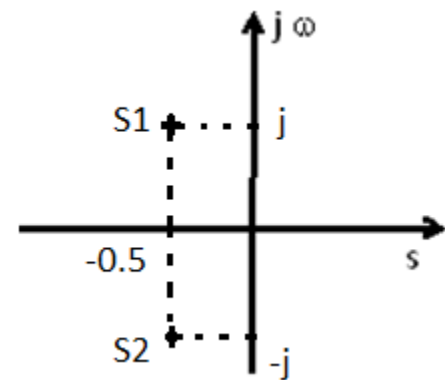
$\omega_n = 1$  rad/sec and  $\zeta = 0.5$  (underdamped).

Now;

$$t_s = \frac{4}{\zeta \omega_n} = 8 \text{ seconds}$$

$$M_p = e^{-\frac{(\zeta \pi)}{\sqrt{1-\zeta^2}}} = e^{-\frac{(\pi)}{\sqrt{0.75}}} = 0.0265 = 2.65\%$$

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} = 1 * \sqrt{1 - (0.5)^2} = 0.866 \text{ rad/sec}$$







$$\text{and } \sigma = \zeta \omega_n = 0.5 * 1 = 0.5$$

$$\text{Also } \beta = \tan^{-1} \left( \frac{\omega_d}{\sigma} \right) = \tan^{-1} \left( \frac{0.866}{0.5} \right) = 1.04$$

Therefore;

$$t_r = \frac{\pi - \beta}{\omega_d} = \frac{\pi - 1.04}{0.866} = 2.42 \text{ seconds}$$

and

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{0.866} = 3.62 \text{ seconds}$$

#### **Example 4.6:**

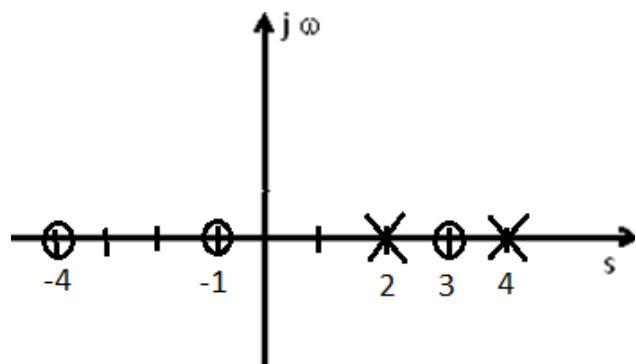
Find the poles and Zeros of the following T.F. and locate on the s-plane.

$$G(s) = \frac{(s^2 + 5s + 4)(s - 3)}{s^2 - 6s + 8}$$

To find the Zeros, set the numerator to Zero.

$$(s^2 + 5s + 4)(s - 3) = 0 \rightarrow (s + 4)(s + 1)(s - 3) = 0$$

then Zeros are  $z_1 = -1$ ,  $z_2 = -4$ , and  $z_3 = 3$ .



To find the Poles, set the denominator to zero.

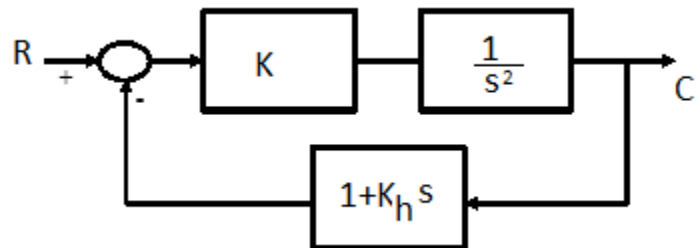


$$s^2 - 6s + 8 = 0 \rightarrow (s - 2)(s - 4) = 0$$

The poles are then,  $p_1=2$  and  $p_2=4$ .

### Example 4.7:

For the given system, determine the values of  $K$  and  $K_h$  so that maximum overshoot to unit step input is 25% and overshoot time equal 2 seconds.



Solution:

Since  $M_p=0.25$ , then

$$\zeta = \frac{\ln^2(0.25)}{\pi^2 + \ln^2(0.25)} = 0.4$$

and

$$t_p = 2 = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}} \rightarrow \omega_n = 1.71 \text{ rad/sec}$$

From the system diagram, the system T.F. is

$$\frac{c(s)}{R(s)} = \frac{\frac{k}{s^2}}{1 + \frac{k}{s^2}(1 + k_h s)} = \frac{k}{s^2 + k + k k_h s}$$

By comparing with the second order standard form, obtaining;

$$k = \omega_n^2 = (1.71)^2 = 2.93$$

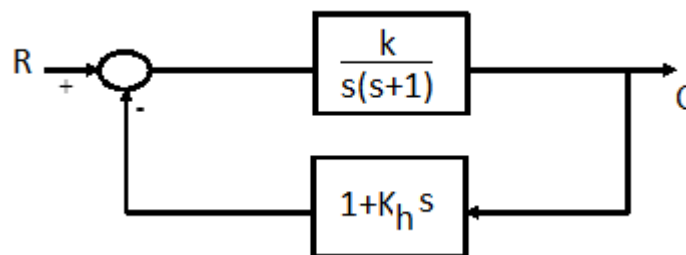
and



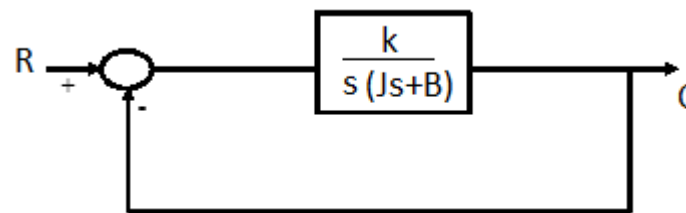
$$k k_h = 2\zeta\omega_n \rightarrow k_h = \frac{2\zeta\omega_n}{k} = \frac{2*0.4*1.71}{2.93} = 0.466$$

Exercises:

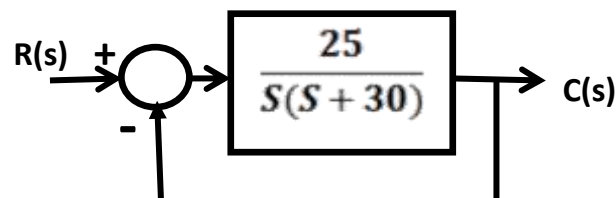
1- For the given system, determine the values of  $K$  and  $K_h$  so that maximum over shoot to unit step input is 0.2 and over shoot time equal 1 seconds. Determine the rise time and settling time.



2- For the control system shown, if  $\zeta=0.6$  and  $\omega_n=5$  rad/sec, determine  $t_p$ ,  $\omega_d$ ,  $t_s$ ,  $t_r$ , and  $M_p$ . Also determine  $J$ ,  $B$ , ( $k=2$ ) for the system when the input is unit step. Also find the final value of output and steady state error and maximum value.

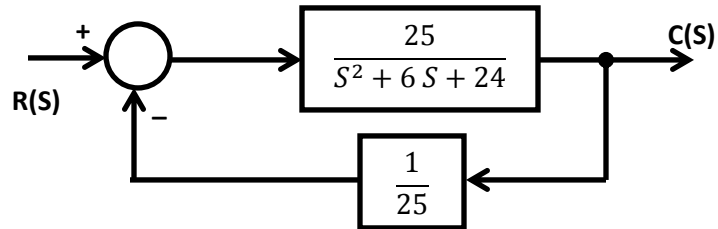


3- Consider the system shown. Find the rise time, peak time, maximum overshoot, and settling time when the system is subjected to a **unit-step** input.





4- For the system shown with unit step input; calculate all time response parameters ( $T_r$ ,  $T_p$ ,  $T_s$ , and  $M_p$ ).



5- Design a second order system with unit step input and peak time equal or less than (0.8 sec ) and maximum overshoot equal or less than ( 6% ). Also calculate time response parameters ( $T_r$ , and  $T_s$ ).