LECTURE FIVE
Stability Criteria

5.1 Routh's Stability Criterion

The most important problem in linear control systems concerns stability. That is, under what conditions will a system become unstable? If it is unstable, how should we stabilize the system? It was stated before that a control system is stable if and only if all closed-loop poles lie in the **left-half s plane**. Most linear closed-loop systems have closed-loop transfer functions of the form:

$$\frac{C(s)}{R(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n} = \frac{B(s)}{A(s)}$$

where the a's and b's are constants and $m \leq n$. A simple criterion, known as Routh's stability criterion, enables us to determine the number of closed-loop poles that lie in the right-half s plane without having to factor the denominator polynomial.

The procedure in Routh's stability criterion is as follows:

1. Write the polynomial of the denominator in (s) in the following form:

$$a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = 0 \tag{5.1}$$

where the coefficients are real quantities. We assume that an Z 0; that is, any zero root has been removed.

2- If any of the coefficients are zero or negative in the presence of at least one positive coefficient, a root or roots exist that are imaginary or that have positive real parts. Therefore, in such a case, the system is not stable. If we are interested in only the absolute stability, there is no need to follow the procedure further. **Note that all the coefficients must be positive**. This is a necessary condition, as may be seen from the following argument: A polynomial in s having real coefficients can always be factored into linear and quadratic factors, such as (s + a) and (s^2 + b s + c), where a, b, and c are real. The linear factors yield real roots and the quadratic factors yield complex-conjugate roots of the polynomial. The factor (s^2+ b s + c) yields roots having negative real parts **only if b and c are both positive**. For all roots to have negative real parts, the constants a, b, c, and so on, in all factors must be positive.

3. If all coefficients are positive, arrange the coefficients of the polynomial in rows and columns according to the following pattern:

$$
\begin{array}{llllll}
s^n & a_0 & a_2 & a_4 & a_6 & \cdots \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
s^{n-4} & d_1 & d_2 & d_3 & d_4 & \cdots \\
\cdot & \cdot & \cdot & & & \\
\cdot & \cdot & \cdot & & & \\
\cdot & \cdot & \cdot & & & \\
s^2 & e_1 & e_2 & & & \\
s^1 & f_1 & & & & \\
s^0 & g_1 & & & &
\end{array}
$$

The process of forming rows continues until we run out of elements. (The total number of rows is n+1). The coefficients b1, b2, b3, and so on, are evaluated as follows:

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}$$

$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}$$

$$b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1}$$

$$\vdots$$

The evaluation of the b's is continued until the remaining ones are all zero. The same pattern of cross-multiplying the coefficients of the two previous rows is followed in evaluating the c's, d's, e's, and so on. That is,

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}$$

$$c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}$$

$$c_3 = \frac{b_1 a_7 - a_1 b_4}{b_1}$$

$$\vdots$$

and

$$d_1 = \frac{c_1 b_2 - b_1 c_2}{c_1}$$

$$d_2 = \frac{c_1 b_3 - b_1 c_3}{c_1}$$

$$\vdots$$

Therefor;

If the system is stable (all Coefficients of the denominator should be positive)

Example 5.1: Apply Routh's stability criterion to find the conditions of selecting the coefficients to ensure stability to a system having the following third-order polynomial:

$$a_0 s^3 + a_1 s^2 + a_2 s + a_3 = 0$$

Solution:

Since all the coefficients are positive numbers. The array of coefficients becomes

$$\begin{array}{lll} s^3 & a_0 & a_2 \\ s^2 & a_1 & a_3 \\ s^1 & \dfrac{a_1 a_2 - a_0 a_3}{a_1} & \\ s^0 & a_3 & \end{array}$$

The condition that all **roots have negative real parts** is given by

$$a_1 a_2 > a_0 a_3$$

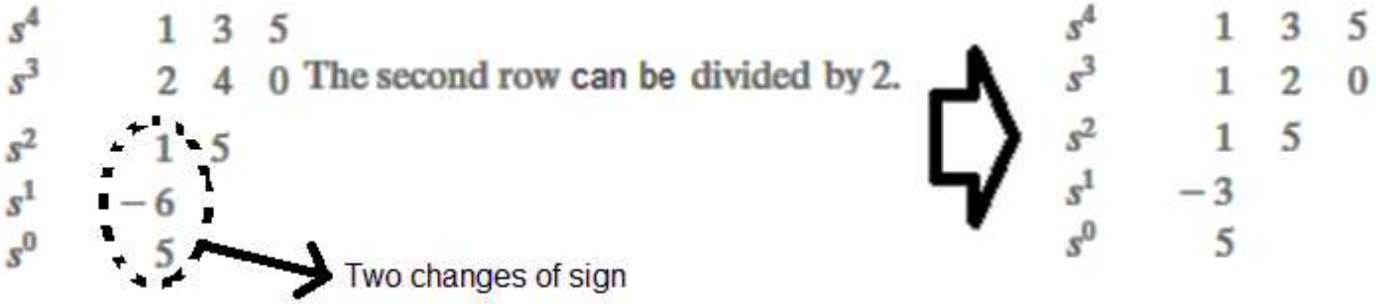
and this is the condition that will ensure stability to the system.

Example 5.2: Apply Routh's stability criterion to the following fourth-order polynomial:

$$s^4 + 2s^3 + 3s^2 + 4s + 5 = 0$$

Solution:

Let us follow the procedure just presented and construct the array of coefficients. (The first two rows can be obtained directly from the given polynomial. The remaining terms are obtained from these. If any coefficients are missing, they may be replaced by zeros in the array).



As it is shown, the number of changes in sign of the coefficients in the first column is 2. <u>This means that there are two roots with positive real parts</u>. Note that the result is unchanged when the coefficients of any row are multiplied or divided by a positive number in order to simplify the computation.

<u>Example 5.3:</u> Apply Routh's to determine the stability of the system with the below characteristic (denominator) equation:

$3\,s^4 + 10\,s^3 + 5\,s^2 + 5\,s + 2 = 0$

Solution:

$\mathbf{S^4}$	**3**	**5**	**2**
$\mathbf{S^3}$	**10**	**5**	
$\mathbf{S^2}$	**3.5**	**2**	
$\mathbf{S^1}$	**-0.714**		
$\mathbf{S^0}$	**2**		

Since there are two sign changes in the first column, then two roots lie on the R.H.S. of s-plane. This means that the system is unstable.

<u>Special Cases:</u> If a first-column term in any row is **<u>zero</u>**, but the remaining terms are not zero or there is no remaining term, then <u>the zero term is replaced by a very small positive number</u> (d) and the rest of the array is evaluated. For example, consider the following equation:

$$s^3 + 2s^2 + s + 2 = 0 \tag{5.2}$$

The array of coefficients is

$$\begin{array}{lll} s^3 & 1 & 1 \\ s^2 & 2 & 2 \\ s^1 & \text{d} - - - - - - - - - \text{d} \approx 0 & \\ s^0 & 2 & \end{array}$$

If the sign of the coefficient above the zero (d) is the same as that below it, it indicates that there are pair of imaginary roots. Actually, Equation (5.2) has two roots at s=± j.

If, however, the sign of the coefficient above the zero (d) is opposite that below it, it indicates that there is <u>one sign change</u>. For example, for the equation

$$s^3 - 3s + 2 = (s - 1)^2(s + 2) = 0$$

the array of coefficients is

$$\begin{array}{llll} \text{One sign change:} & s^3 & 1 & -3 \\ & s^2 & \text{d} & 2 \\ & s^1 & -3 - \dfrac{2}{\text{d}} & \\ \text{One sign change:} & s^0 & 2 & \end{array} \qquad \text{d} \approx 0$$

There are two sign changes of the coefficients in the first column. So there are two roots in the right-half s plane. This agrees with the correct result indicated by the factored form of the polynomial equation.

Example 5.4: Apply Routh's to determine the stability of the system with the below characteristic (denominator) equation:

$s^5 + s^4 + 2\,s^3 + 2\,s^2 + 3\,s + 5 = 0$

Solution:

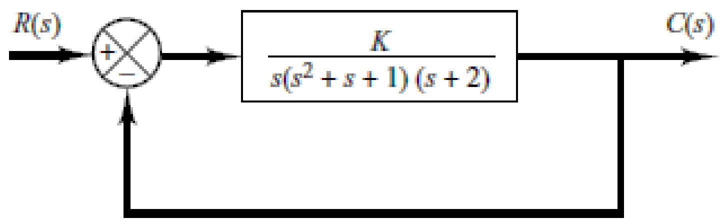
S^5	1	2	3
S^4	1	2	5
S^3	d	-2	
S^2	$\frac{2d+2}{d}$	5	
S^1	$\frac{-4d-4-5d^2}{2d+2}$		
S^0	5		

Therefore, the system is unstable because there are two changes in sign in first column.

5.2 Application of Routh's Stability Criterion to Control-System Analysis

Routh's stability criterion is of limited usefulness in linear control-system analysis, mainly because it does not suggest how to improve relative stability or how to stabilize an unstable system. It is possible, however, to determine the effects of changing one or two parameters of a system by examining the values that cause instability. In the following, we shall consider the problem of determining the stability range of a parameter value.

Example 5.5: For the system shown below, determine the range of (K) that ensures system stability.



Solution:

The closed loop transfer function is:

$$\frac{C(s)}{R(s)} = \frac{K}{s(s^2 + s + 1)(s + 2) + K}$$

The characteristic equation is

$$s^4 + 3s^3 + 3s^2 + 2s + K = 0$$

The array of coefficients becomes

$$\begin{array}{lccc} s^4 & 1 & 3 & K \\ s^3 & 3 & 2 & 0 \\ s^2 & \frac{7}{3} & K & \\ s^1 & 2 - \frac{9}{7}K & & \\ s^0 & K & & \end{array} \quad \Rightarrow \quad \begin{array}{lcccl} s^4 & 1 & 3 & K & \leftarrow \text{-- Should be > 0} \\ s^3 & 3 & 2 & 0 & \\ s^2 & \frac{7}{3} & K & & \leftarrow \text{-- Should be > 0} \\ s^1 & 2 - \frac{9}{7}K & & & \leftarrow \text{---- Should be > 0} \\ s^0 & K & & & \leftarrow \text{---- Should be > 0} \end{array}$$

For stability, (K) must be positive, and all coefficients in the first column must be positive. Therefore,

$$\frac{14}{9} > K > 0$$

When k=14/9 the system becomes <u>oscillatory</u> and, mathematically, the oscillation is sustained at constant amplitude.

Example 5.6: For the system with characteristic equation below, determine the range of (K) for which the system is stable.

$$S^3 + 3\,K\,S^2 + KS + 2\,S + 4 = 0$$

Solution:

Rearranging the equation;

$S^3 + 3\,K\,S^2 + (K+2)\,S + 4 = 0$

Applying Routh's array as below;

S^3	1	(K+2)
S^2	3K	4
	$\frac{3K(K+2)-4}{3K}$	
S^0	4	

Now to ensure system stability, below conditions must be satisfied;

also;

$$\frac{3K(K+2)-4}{3K} > 0 \quad \rightarrow \quad 3K(K+2)-4 > 0 \quad \rightarrow \quad 3K^2+6K-4 > 0$$

$K^2+2K-1.33 > 0$ this will give two values of K;

either K > (-1+1.526) > 0.526 or K > (-1-1.526) > -2.526.

and since K is positive, then for stability;

$$\mathbf{K > 0.526}$$

Example 5.7: For the system with characteristic equation below, determine the range of (K) for which the system is stable.

$$S^4 + 20\,K\,S^3 + 5\,S^2 + 10\,S + 15 = 0$$

Solution:

Applying Routh's array as below;

S^4	1	5	15
S^3	20K	10	
S^2	$5 - \frac{1}{2K}$	15	
S^1	$10 - \frac{600\,K^2}{10\,K - 1}$		
S^0	15		

Now; first of all **K>0**

And from 3rd expression in the first column,

$$5 - \frac{1}{2K} > 0 \;\rightarrow\; \frac{1}{2K} < 5 \;\rightarrow\; 2K > 0.2 \;\rightarrow\; \boldsymbol{K > 0.1}$$

While the 4th expression in the first column gives;

$$10 - \frac{600\,K^2}{10\,K - 1} > 0 \;\rightarrow\; 10 > \frac{600\,K^2}{10\,K - 1} \;\rightarrow\; 600\,K^2 > 100\,K - 10$$

$$600\,K^2 - 100\,K + 10 < 0$$

This will give **K=0.083 ± j 0.098**

And since K is complex number, the system is unstable

Exercises:

For the system with characteristic equations below, determine the range of (K) for which the system is stable.

1- $S^4 + 20 S^3 + 224 S^2 + 1240 S + 2400 + K = 0$

2- $S^3 + (K+0.5) S^2 + 4K S + 50 = 0$

3- $S^4 + 4 S^3 + 4 S^2 + 3 S + K = 0$

4- $S^3 + 8 S^2 + 15 S + K S + 2K = 0$

5- $0.1 S^3 + 0.25 S^2 + 2 S + K = 0$

6- $(K+1) S^2 + (3K-0.9) S + (2K-0.1) = 0$

7- $S^3 + 3K S^2 + (K+2) S + 4 = 0$

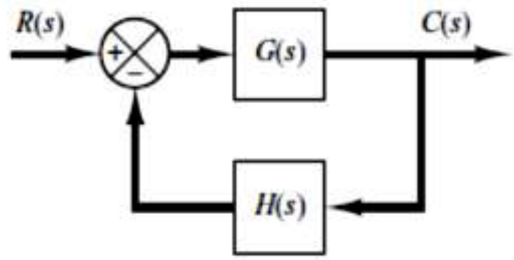
8- $S^3 + 10 S^2 + (21+K) S + 13 K = 0$

9- $S^4 + 12 S^3 + 69 S^2 + 198 S + (200+K) = 0$

5.3 Control Systems Analysis by the Root-Locus Method

ROOT-LOCUS PLOTS :

Angle and Magnitude Conditions: Consider the negative feedback system shown in figure below. The closed-loop transfer function is;



$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

The characteristic equation for this closed-loop system is

$$1 + G(s)H(s) = 0 \quad \text{or} \quad G(s)H(s) = -1$$

Here we assume that **G(s)H(s)** is a ratio of polynomials in s. Since **G(s)H(s)** is a complex quantity, equation above can be split into two equations by equating the **angles and magnitudes** of both sides, respectively, to obtain the following:

Angle condition:

$$\boxed{\angle G(s)H(s) = \pm 180^\circ(2k + 1) \qquad (k = 0, 1, 2, \ldots)} \tag{5.3}$$

Magnitude condition:

$$\boxed{|G(s)H(s)| = 1} \tag{5.4}$$

The values of (s) that fulfil both the angle and magnitude conditions are the roots of the characteristic equation, or the closed-loop poles. A locus of the points in the complex plane satisfying the **angle condition alone is the root locus**. The **roots** of the characteristic equation (the closed-loop poles) corresponding to a given value of the gain **can be determined from the magnitude condition**.

In many cases, G(s)H(s) involves a gain parameter K, and the characteristic equation may be written as

$$1+\frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}=0$$

Then **the root loci for the system are the loci of the closed-loop poles as the gain *K* is varied from zero to infinity.** Note that to begin sketching the root loci of a system by the root-locus method we must know the location of the poles and zeros of G(s)H(s). Remember that the angles of the complex quantities originating from the open-loop poles and open-loop zeros to the test point s are measured in the counter clockwise direction. For example, if G(s)H(s)is given by

$$G(s)H(s)=\frac{K(s+z_1)}{(s+p_1)(s+p_2)(s+p_3)(s+p_4)}$$

where –p2 and –p3 are complex-conjugate poles, then the angle of G(s)H(s) is

$$\boxed{\underline{/G(s)H(s)}=\phi_1-\theta_1-\theta_2-\theta_3-\theta_4}$$

where ф1 , ɵ1 , ɵ2 , ɵ3 , and ɵ4 are measured counter clockwise as shown in Figures 5.1(a) and (b).The magnitude of G(s)H(s) for this system is

$$\boxed{|G(s)H(s)|=\frac{KB_1}{A_1A_2A_3A_4}}$$

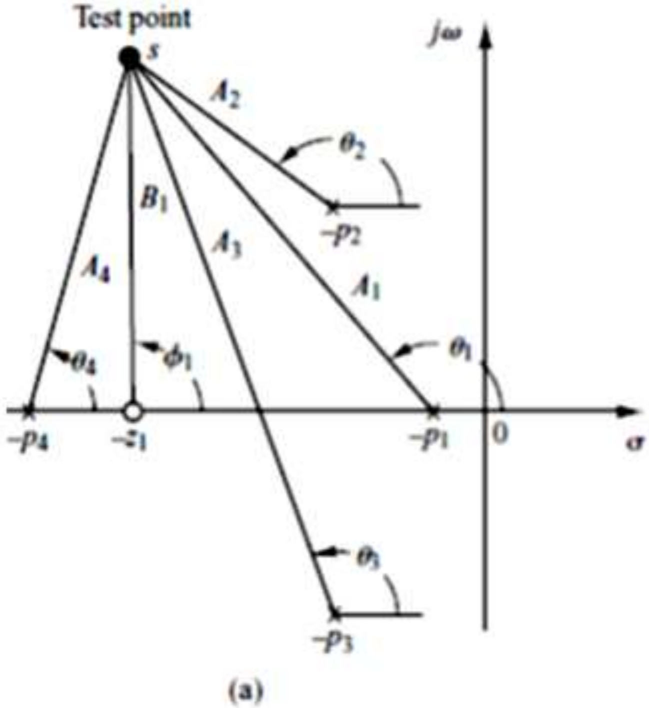


Figure 5.1 (a) and (b) Diagrams showing angle measurements from open-loop poles and open-loop zero to test point s.

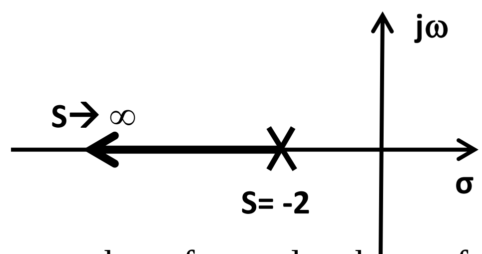
where $A1$, $A2$, $A3$, $A4$, and $B1$ are the magnitudes of the complex quantities s+p1 , s+p2, s+p3, s+p4 , and s+z1 , respectively.

In what follows, two illustrative examples for constructing root-locus plots will be presented. Although computer approaches to the construction of the root loci are easily available, here we shall use graphical computation, combined with inspection, to determine the root loci upon which the roots of the characteristic equation of the closed-loop system must lie. Such a graphical approach will enhance understanding of how the closed-loop poles move in the complex plane as the open loop poles and zeros are moved.

<u>Example 5.7:</u> Plot the root locus pattern of a system whose forward path transfer function is:

$$G(s) = \frac{200}{s+20}$$

Solution: Since there is one pole at s= -20, the plot then starts at s= -20 on the real axis. Furthermore, since this pole terminates at one zero at s→∞ , then the root locus terminates at ∞ from pole s= -2 as shown in the following root locus plot.

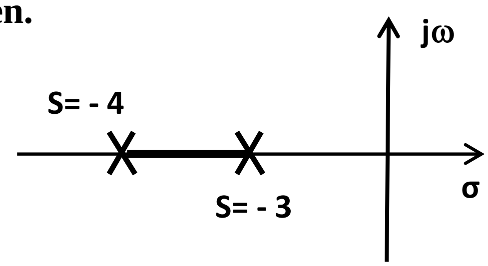


<u>Example 5.8:</u> Plot the root locus pattern of a system whose forward path transfer function is:

$$G(s) = \frac{K}{(S+3)\,(S+4)}$$

Solution: There are two poles at s= -3 and s= -4 and no zero exist. These poles have been located as shown in the following figure. Since there existing two poles, there are two root-loci both terminating at (∞) . Now taking test point between s= -2 and s= -4, we find that the root locus exist between these poles **because the sum of poles and zeros to the right hand side is odd.**

While if the test point is between s= -4 and ∞**,** root locus found to not existing **because the sum of poles and zeros to the right hand side is even.**



<u>Breakaway points may be found as;</u>

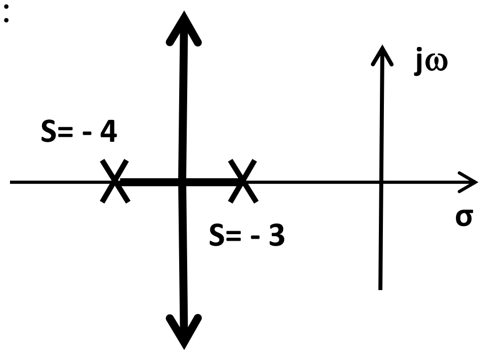
$\frac{d}{ds}G(s) = 2\,s + 7 = \;0 \;\rightarrow\;\; \boldsymbol{s = -3.5}$

Which means that the root locus will depart (breakout) at the half the distance between the poles s=-3 and s=-4.

<u>The Asymptotes to the root loci at infinity may be found as;</u>

$$\propto = \pm\frac{180(2k+1)}{2-0} = \pm 90^{\,0}, \pm 270^{0}$$

Therefore, the complete root-locus is shown below :

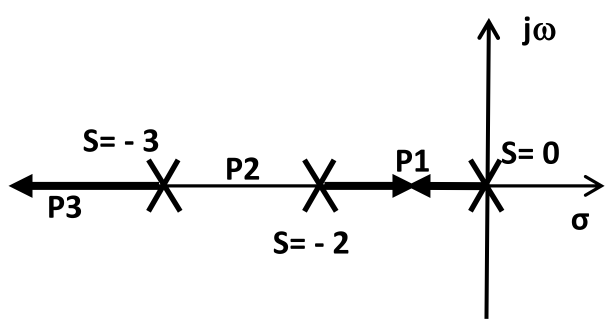


<u>Example 5.9:</u> Plot the root locus pattern of a system whose forward path transfer function is:

$$G(s) = \frac{K}{S(S+2)\,(S+3)}$$

Solution:

There are 3 poles at s=0, s= -2, and s=-3 and no zeros. Since there are 3 poles, there are three root loci. Now to determine which of these 3 root loci are on the root locus and which is not, we should select test points P1 (between s=0 and s= -2), P2 (between s= -2 and s= -3), and P3 (between s= -3 and s→∞) as shown in the figure below;



Now since P1and P3 have ODD number of poles and zeros to their right side, then P1 and P3 are part of ROOT-LOCUS. On the other hand, P2 is not part of the ROOT LOCUS because it has an even number of poles and zeros to its right side.

Breakaway from the real axis :

Characteristic equation is

$s^3 + 5\,s^2 + 6s = 0$

then $\frac{dG(s)}{ds} = 3\,s^2 + 10s + 6 = 0$

and this will give s= -0.784 and s= -2.549

Now, since there is no root locus between s= -2 and s= -3, therefore **s= -0.784** is the breakaway point.

Asymptotes to the root-loci:

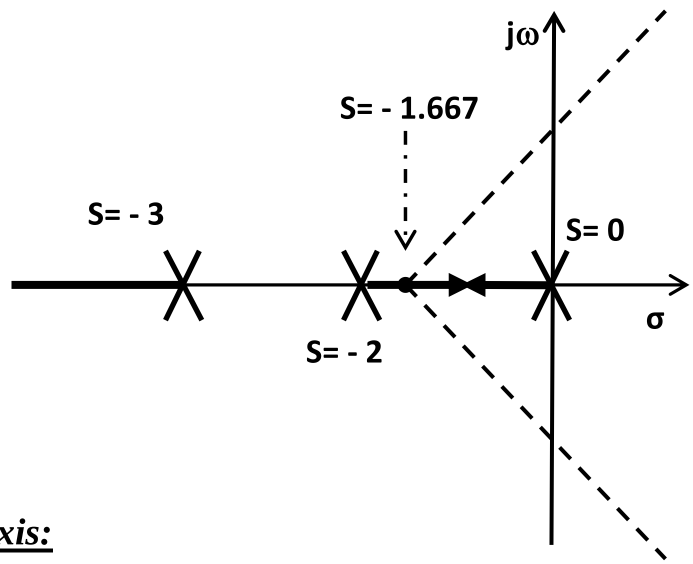
$$\alpha_{1\,,2} = \pm\frac{180}{3} = \pm 60^0$$

$$\alpha_3 = \pm\frac{3 *\ 180}{3} = \pm 180^0$$

Intersection of the Asymptotes on real axis:

$$s = \frac{\sum Poles + \sum Zeros}{Number\ of\ Poles - Number\ of\ Zeros} = \frac{(0-2-3)-(0)}{3-0} = -1.667$$

The three asymptotes with centre of (s= -1.667) making angles ±60 and ± 180 are shown in figure below;



Intersection with Imaginary axis:

$$G(j\omega) = \frac{K}{j\omega(j\omega+2)\,(j\omega+3)} = \frac{K}{-j\omega^3 - 5\,\omega^2 + 6j\omega} = \frac{K}{-j\omega^3 - 5\,\omega^2 + 6j\omega}$$

$$G(j\omega) = \frac{K}{-5\,\omega^2 + j(6\omega - \omega^3)} * \frac{-5\,\omega^2 - j(6\omega - \omega^3)}{-5\,\omega^2 - j(6\omega - \omega^3)} = \frac{-5\,\omega^2 - j(6\omega - \omega^3)}{25\,\omega^4 - (6\omega - \omega^3)^2}$$

$$G(j\omega) = \frac{-5\,\omega^2}{25\,\omega^4 - (6\omega - \omega^3)^2} - j\frac{(6\omega - \omega^3)}{25\,\omega^4 - (6\omega - \omega^3)^2}$$

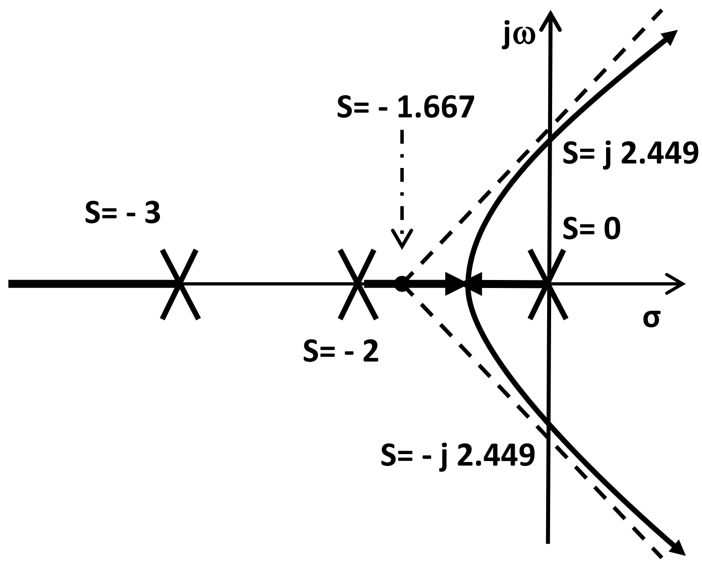
Now equating the imaginary part to zero, get;

$$-\frac{(6\omega - \omega^3)}{25\,\omega^4 - (6\omega - \omega^3)^2} = 0 \quad \rightarrow \quad \omega\,(6 - \omega^2) = 0$$

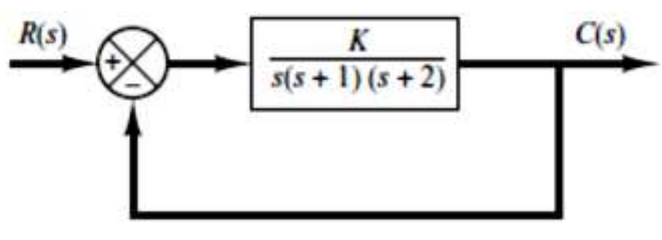
therefore,

$$\omega = \pm\sqrt{6} = \pm 2.449$$

The complete root – locus pattern is shown in the figure below;



Example 5.10: Consider the negative feedback system shown in figure below. (We assume that the value of gain K is nonnegative.) For this system,



$$G(s) = \frac{K}{s(s+1)(s+2)}, \qquad H(s) = 1$$

Let us sketch the root-locus plot and then determine the value of K such that the damping ratio ζ of a pair of dominant complex-conjugate closed-loop poles is 0.5.

For the given system, the angle condition becomes

$$\underline{/G(s)} = \underline{\Big/ \frac{K}{s(s+1)(s+2)}} = -\underline{/s} - \underline{/s+1} - \underline{/s+2} = \pm 180^\circ(2k+1) \qquad (k = 0, 1, 2, \dots)$$

The magnitude condition is

$$|G(s)| = \left| \frac{K}{s(s+1)(s+2)} \right| = 1$$

Determine the root loci on the real axis: The first step in constructing a root-locus plot is to locate the **open-loop poles**, s=0, s= –1, and s= –2, in the complex plane. (There are no open loop zeros in this system.) The locations of the open-loop poles are indicated by **crosses,** while the locations of the open-loop zeros will be indicated by **small circles**).

Note that the **starting points** of the root loci (the points corresponding to K=0) are **open-loop poles**. The number of individual **root loci for this system is three**, which is the same as the **number of open-loop poles**. To determine the root loci on the real axis, we select a test point, s. If (s) is between s=0 and s= -1, then the test point has an odd number of poles and zeros on its right side. Also it can be found by (another way);

$$\underline{/s} = 180^\circ, \qquad \underline{/s+1} = \underline{/s+2} = 0^\circ$$

Thus;

$$-\underline{/s} - \underline{/s+1} - \underline{/s+2} = -180^\circ$$

and the angle **condition is satisfied**. Therefore, the portion of the negative real axis between **0 and –1** forms a portion of the **root locus**. If a test point is selected between –1 and –2, then

$$\underline{/s} = \underline{/s+1} = 180^\circ, \qquad \underline{/s+2} = 0^\circ$$

and

$$-\underline{/s} - \underline{/s+1} - \underline{/s+2} = -360^\circ$$

It can be seen that the angle **condition is not satisfied**. However the test point here has even number of poles and zeros on its right side, therefore, the negative real axis from –1 to –2 is **not a part of the root locus**. Similarly, if a test point is located on the negative real axis **from –2 to –∞, the angle condition is satisfied and the test point here has odd number of poles and zeros on its right side**. Thus, **root loci exist on the negative real axis between 0 and –1 and between –2 and –∞.**

Determine the asymptotes of the root loci: The asymptotes of the root loci as s approaches infinity can be determined as follows: If a test point s is selected very far from the origin, then

$$\text{Angles of asymptotes} = \frac{\pm 180^\circ(2k+1)}{3} \qquad (k = 0, 1, 2, \dots)$$

$$\boxed{\begin{array}{l} \text{Angles of asymptotes} = \dfrac{\pm 180^\circ(2k+1)}{n-m} \qquad (k = 0, 1, 2, \dots) \\ \text{where } n = \text{number of finite poles of } G(s)H(s) \\ \qquad\;\; m = \text{number of finite zeros of } G(s)H(s) \end{array}}$$

Since the angle repeats itself as K is varied, the distinct angles for the asymptotes are determined as **60°, –60°, and 180°.** Thus, there are **<u>three asymptotes</u>**. The one having the angle of 180° is the negative real axis.

Before we can draw these asymptotes in the complex plane, we **must find the point where they <u>intersect</u>** the **<u>real axis</u>**.

$$s = \frac{\sum Poles + \sum Zeros}{Number\ of\ Poles - Number\ of\ Zeros} = \frac{(0-1-2)-(0)}{3-0} = \mathbf{-1}$$

Intersection with imaginary axis:

$$G(j\omega) = \frac{K}{j\omega(j\omega+1)\,(j\omega+2)} = \frac{K}{-j\omega^3 - 3\omega^2 + j2\omega}$$

$$= \frac{K}{-3\omega^2 + j(2\omega - \omega^3)} * \frac{-3\omega^2 - j(\omega - \omega^3)}{-3\omega^2 - j(2\omega - \omega^3)}$$

$$G(j\omega) = \frac{K(-3\omega^2 - j(2\omega - \omega^3))}{9\omega^4 + (2\omega - \omega^3)^2} = 0$$

Equating the imaginary part to zero, results in;

$\boldsymbol{\omega = \pm\sqrt{2}}$

These points can also be found by use of Routh's stability criterion as follows: Since the characteristic equation for the present system is

$$s^3 + 3s^2 + 2s + K = 0$$

the Routh array becomes

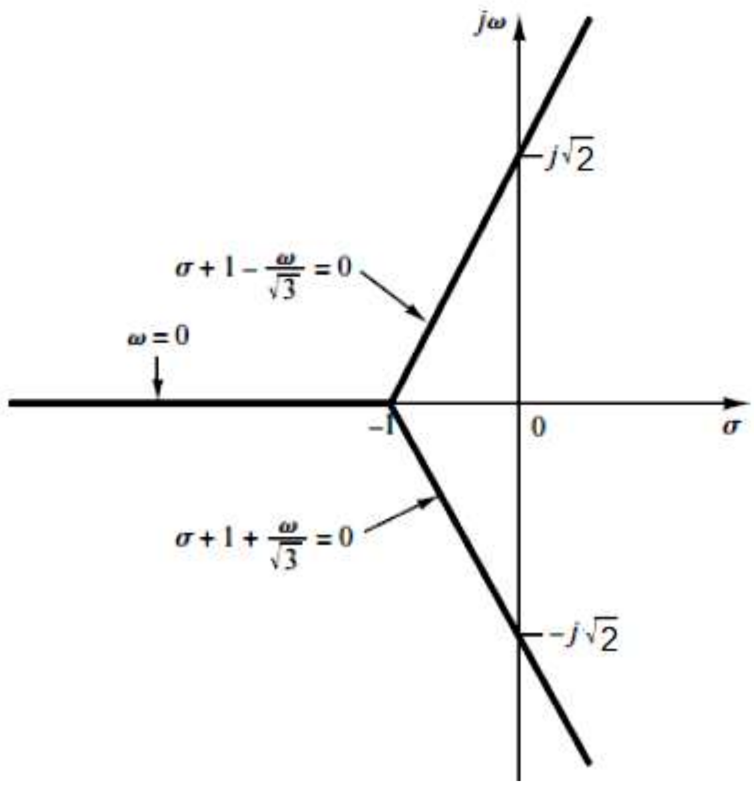
$$\begin{array}{lcc} s^3 & 1 & 2 \\ s^2 & 3 & K \\ s^1 & \dfrac{6-K}{3} & \\ s^0 & K & \end{array}$$

The value of K that makes the s1 term in the first column equal zero is K=6 .The crossing points on the imaginary axis can then be found by solving the auxiliary equation obtained from the s2 row; that is,

$$3s^2 + K = 3s^2 + 6 = 0$$

which yields;

$$s = \pm j\sqrt{2}$$



Determine the breakaway point: To plot root loci accurately, we must find the breakaway points, where the root-locus branches originating from the poles at 0 and –1 break away (as K is increased) from the real axis and move into the complex plane. The breakaway point corresponds to a **point in the s plane where multiple roots of the characteristic equation occur**. A simple method for finding the breakaway point is available. We shall present this method in the following: Let us write the characteristic equation as

$$\boxed{f(s) = B(s) + KA(s) = 0}$$

where A(s) and B(s) do not contain K. Note that f(s)=0 has multiple roots at points where

$$\boxed{\frac{df(s)}{ds} = 0}$$

Fter a series of derivations and substitutions, the breakaway points can be simply determined from the roots of:

$$\boxed{\frac{dK}{ds} = 0}$$

For the present example, the characteristic equation G(s)+1=0 is given by

$$\frac{K}{s(s+1)(s+2)} + 1 = 0 \quad \text{or} \quad K = -(s^3 + 3s^2 + 2s)$$

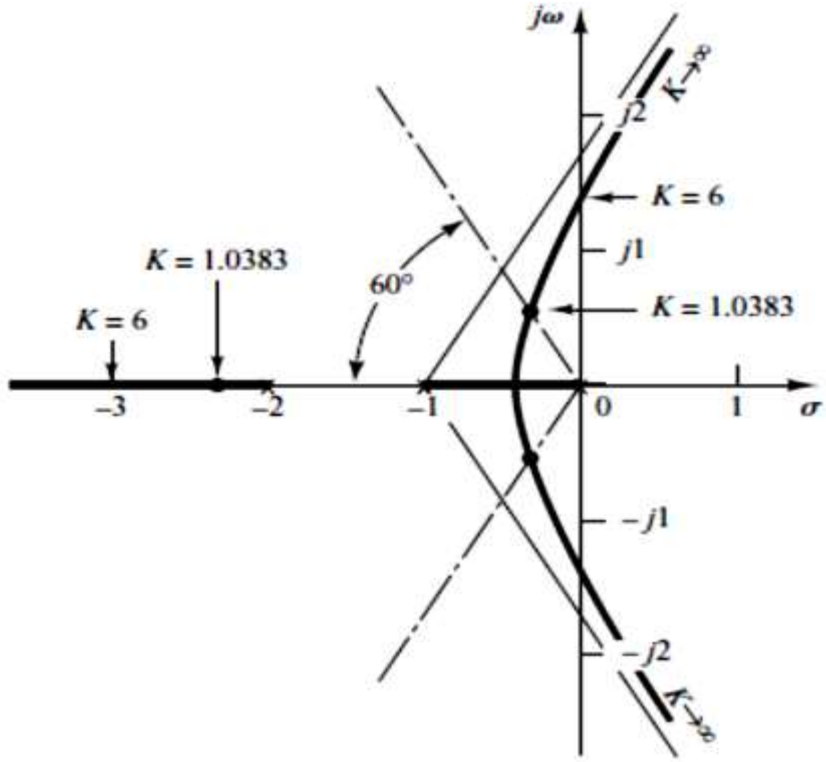
By setting *dK*/*ds*=0, we obtain

$$\frac{dK}{ds} = -(3s^2 + 6s + 2) = 0 \quad \text{or} \quad s = -0.4226, \qquad s = -1.5774$$

Since the breakaway point must lie on a root locus between 0 and –1, it is clear that *s*=–0.4226 corresponds to the actual breakaway point. Point s= –1.5774 is not on the root locus. Hence, this point is not an actual breakaway or break-in point. In fact, evaluation of the values of K corresponding to s= –0.4226 and s= –1.5774 yields:

$$K = 0.3849, \qquad \text{for } s = -0.4226$$
$$K = -0.3849, \qquad \text{for } s = -1.5774$$

Draw the root loci, based on the information obtained in the foregoing steps, as shown in figure below.



Determine a pair of dominant complex-conjugate closed-loop poles such that the damping ratio ζ is 0.5. Closed-loop poles with ζ=0.5 lie on lines passing through the origin and making the angles $\pm\cos^{-1}\zeta = \pm\cos^{-1}0.5 = \pm 60^\circ$ with the negative real axis. From last figure, such closed loop poles having ζ=0.5 are obtained as follows:

$$s_1 = -0.3337 + j0.5780, \qquad s_2 = -0.3337 - j0.5780$$

The value of K that yields such poles is found from the magnitude condition as follows:

$$\begin{aligned} K &= |s(s+1)(s+2)|_{s=-0.3337+j0.5780} \\ &= 1.0383 \end{aligned}$$

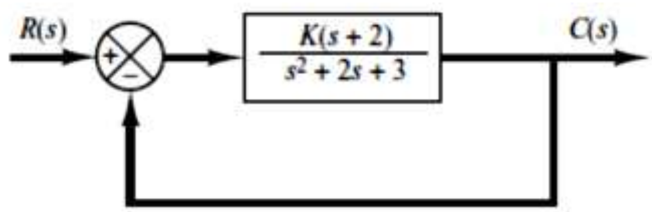
Using this value of K, the third pole is found at s=–2.3326.

Note that, from step 4, it can be seen that for (K=6) the dominant closed-loop poles lie on the imaginary axis at $s = \pm j\sqrt{2}$. With this value of K, the system will exhibit sustained oscillations. For K>6, the dominant closed-loop poles lie in the right-half s plane, resulting in an unstable system. Finally, note that, if necessary, the root loci can be easily graduated in terms of (K) by use of the magnitude condition. We simply pick out a point on a root locus, measure the magnitudes of the three complex quantities s, s+1, and s+2, and multiply these magnitudes; the product is equal to the gain value K at that point, or

$$|s| \cdot |s+1| \cdot |s+2| = K$$

Example 5.11: In this example, we shall sketch the root-locus plot of a system with complex-conjugate open-loop poles. Consider the negative feedback system shown in figure below. For this system,

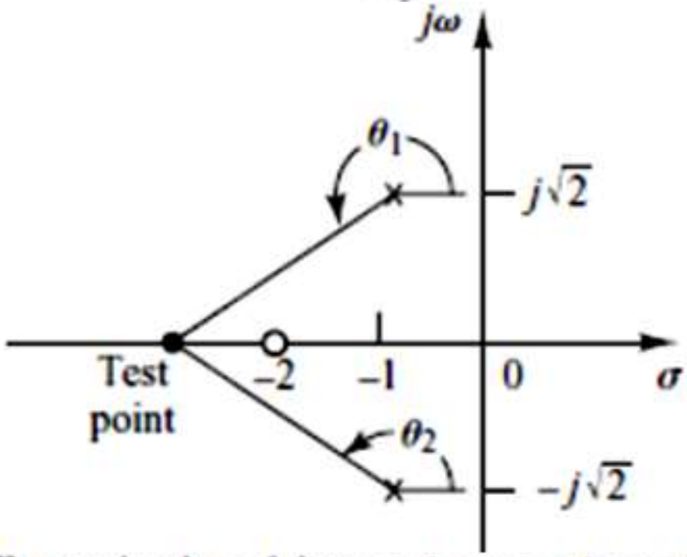
$$G(s) = \frac{K(s+2)}{s^2+2s+3}, \qquad H(s) = 1$$



where $K \geq 0$. It is seen that G(s) has a pair of complex-conjugate poles at

$$s = -1 + j\sqrt{2}, \qquad s = -1 - j\sqrt{2}$$

***Determine the root loci on the real axis**:* For any test point s on the real axis, the sum of the angular contributions of the complex-conjugate poles is 360°, as shown in the following figure.



Determination of the root locus on the real axis.

Thus the net effect of the complex-conjugate poles is zero on the real axis. The location of the root locus on the real axis is determined from the open-loop zero on the negative real axis. A simple test reveals that a section of the negative real axis, that between –2 and – ∞, is a part of the root locus. It is noted that, since this locus lies between two zeros (at s= –2 and s= –∞), it is actually a part of two root loci, each of which starts from one of the two complex-conjugate poles. In other words, two root loci break in the part of the negative real axis between –2 and – ∞. Since there are two open-loop poles and one zero, there is **one asymptote**, which coincides with the negative real axis.

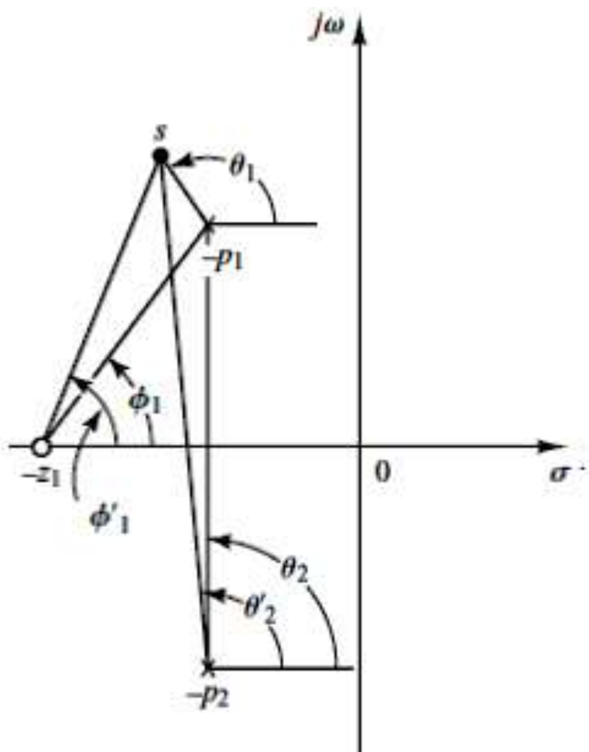
Determine the angle of departure from the complex-conjugate open-loop poles: The presence of a pair of complex-conjugate open-loop poles requires the determination of the angle of departure from these poles. Knowledge of this angle is important, since the root locus near complex pole yields information as to whether the locus originating from the complex pole migrates toward the real axis or extends toward the asymptote.

Referring to figure below, if we choose a test point and move it in the very vicinity of the complex open-loop pole at s= –p1, we find that the sum of the angular contributions from the pole at s= p2 and zero at s= –z1 to the test point can be considered remaining the same. If the test point is to be on the root locus, then the sum of $\phi'_1, -\theta_1$, and $-\theta'_2$ must be $\pm 180^\circ(2k+1)$, where k=0, 1, 2,p .Thus, in the example,

$\phi'_1 - (\theta_1 + \theta'_2) = \pm 180^\circ(2k+1)$ or $\theta_1 = 180^\circ - \theta'_2 + \phi'_1 = 180^\circ - \theta_2 + \phi_1$

The angle of departure is then

$\theta_1 = 180^\circ - \theta_2 + \phi_1 = 180^\circ - 90^\circ + 55^\circ = 145^\circ$



Determination of the angle of departure.

Since the root locus is symmetric about the real axis, the angle of departure from the pole at $s = -p_2$ is $-145°$.

3. ***Determine the break-in point:*** A break-in point exists where a pair of root-locus branches meets as K is increased. For this problem, the break-in point can be found as follows: Since

$$K = -\frac{s^2 + 2s + 3}{s + 2}$$

We have

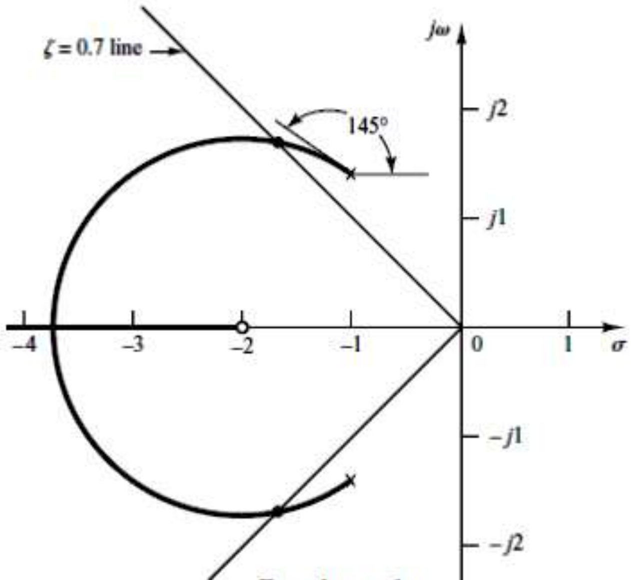
$$\frac{dK}{ds} = -\frac{(2s + 2)(s + 2) - (s^2 + 2s + 3)}{(s + 2)^2} = 0$$

Which gives

$s^2 + 4s + 1 = 0$ or $s = -3.7320$ or $s = -0.2680$

Notice that point s= –3.7320 is on the root locus. Hence this point is an actual break-in point. (Note that at point s= –3.7320 the corresponding gain value is K=5.4641.) Since point s= –0.2680 is not on the root locus, it cannot be a break-in point. (For point s= –0.2680, the corresponding gain value is K= –1.4641).

4. ***Sketch a root-locus plot, based on the information obtained in the foregoing steps*****:** To determine accurate root loci, several points must be found by trial and error between the breaking point and the complex open-loop poles. (To facilitate sketching the root-locus plot, we should find the **direction in which the test point should be moved** by mentally summing up the changes on the angles of the poles and zeros). Figure below shows a complete root-locus plot for the system considered.



Root-locus plot.

The value of the gain K at any point on root locus can be found by applying the magnitude Condition. For example, the value of *K* at which the complex-conjugate closed-loop poles have the damping ratio ζ=0.7 can be found by locating the roots, on root locus plot figure, and computing the value of *K* as follows:

$$K = \left| \frac{(s+1-j\sqrt{2})(s+1+j\sqrt{2})}{s+2} \right|_{s=-1.67+j1.70} = 1.34$$