BODE DIAGRAMS

6.1 Bode Diagrams or Logarithmic Plots:

A Bode diagram consists of **two graphs**: One is a plot of the logarithm of the **magnitude of a sinusoidal transfer function**; the other is a plot of the **phase angle**; both are plotted against the frequency on a **logarithmic Scale**. The standard representation of the logarithmic magnitude of G(jω) is **20 log |G(jω)|,** where the base of the logarithm is **10**.The unit used in this representation of the magnitude is the decibel, usually abbreviated **dB**. In the logarithmic representation, the curves are drawn on **semilog paper**, using the log scale for frequency and the linear scale for either **magnitude (but in decibels) or phase angle (in degrees).** (The frequency range of interest determines the number of logarithmic cycles required on the abscissa.).

The main advantage of using the Bode diagram is that multiplication of magnitudes can be converted into addition. Furthermore, a simple method for sketching an approximate log-magnitude curve is available. It is based on **asymptotic approximations**. Such approximation by straight-line asymptotes is sufficient if only rough information on the frequency-response characteristics is needed. Should the exact curve be desired, corrections can be made easily to these basic asymptotic plots. Expanding the low-frequency range by use of a logarithmic scale for the frequency is highly advantageous, since characteristics at low frequencies are most important in practical systems. Although it is not possible to plot the curves right down to zero frequency because of the logarithmic frequency **(log0= –∞),** this does not create a serious problem. Note that the experimental determination of a transfer function can be made simple if frequency-response data are presented in the form of a Bode diagram.

6.2 Basic Factors of G(jω) H(jω).

As stated earlier, the main advantage in using the logarithmic plot is the relative ease of plotting frequency-response curves. The basic factors that very frequently occur in an arbitrary transfer function G(jω)H(jω) are:

1. Gain K

2. Integral and derivative factors (jω)<1

3. First-order factors (1+jωT)<1

4. Quadratic factors

Once we become familiar with the logarithmic plots of these basic factors, it is possible to utilize them in constructing a composite logarithmic plot for any general form of G(jω)H(jω) by sketching the curves for each factor and adding individual curves graphically, because adding the logarithms of the gains corresponds to multiplying them together.

6.2.1 The Gain *K*

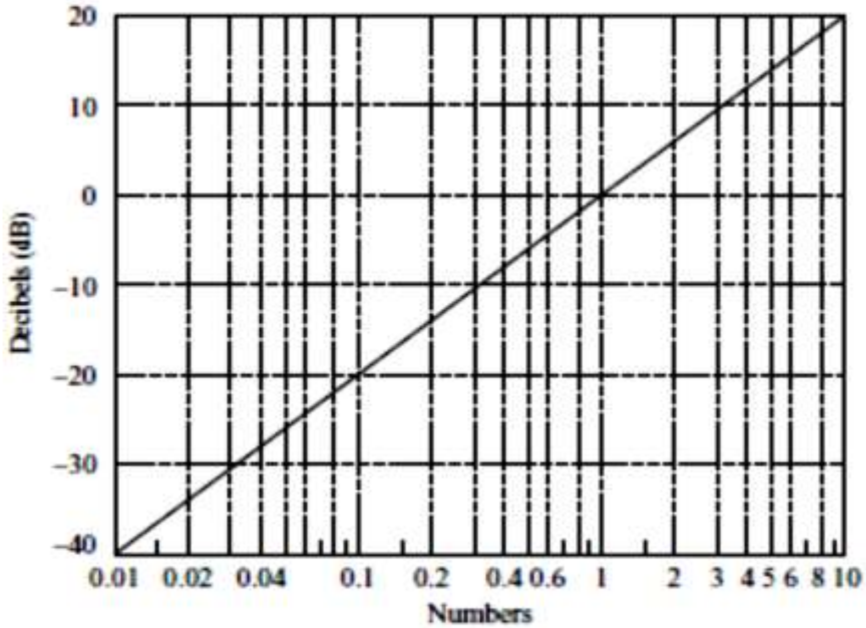
A number greater than unity has a **positive** value in decibels, while a number smaller than unity has a **negative** value. The log-magnitude curve for a constant gain K is a horizontal straight line at the magnitude of " **20 log(K) "** decibels. The phase angle of the **gain K is zero**. The effect of varying the gain K in the transfer function is that it raises or lowers the log-magnitude curve of the transfer function by the corresponding constant amount, but it has no effect on the phase curve. A number–decibel conversion line is given in Figure below.

The decibel value of any number can be obtained from this line. As a number increases by a factor of 10, the corresponding decibel value increases by a factor of 20. This may be seen from the following:

$$20\log(K \times 10) = 20\log K + 20$$

Similarly,

$$20\log(K \times 10^n) = 20\log K + 20n$$



Note that, when expressed in decibels, the reciprocal of a number differs from its value only in **sign**; that is, for the number K,

$$20\log K = -20\log\frac{1}{K}$$

6.2.2 Integral and Derivative Factors $(j\omega)^{\pm 1}$.

The logarithmic magnitude of 1/jω in decibels is

$$20\log\left|\frac{1}{j\omega}\right| = -20\log\omega \text{ dB}$$

The phase angle of 1/jω is constant and equal to –90°.

In Bode diagrams, frequency ratios are expressed in terms **of octaves** or **decades**. An **octave is a frequency band from ω1 to 2ω1 ,where ω1 is any frequency value**. **A decade is a frequency band from ω1 to 10ω1 ,where again ω1 is any frequency**.

(On the logarithmic scale of semilog paper, any given frequency ratio can be represented by the same horizontal distance. For example, the horizontal distance from ω=1 to ω=10 is equal to that from ω=3 to ω=30.)

If the log magnitude **–20 logω** dB is plotted against ω on a logarithmic scale, it is a straight line. To draw this straight line, we need to locate one point (0 dB, ω=1) on it. Since

$$(-20\log 10\omega)\ \text{dB} = (-20\log\omega - 20)\ \text{dB}$$

the slope of the line is –20 dB/decade (or –6 dB/octave). Similarly, the log magnitude of jω in decibels is

$$20\log|j\omega| = 20\log\omega\ \text{dB}$$

The phase angle of jω is constant and equal to 90°. The log-magnitude curve is a straight line with a slope of 20 dB/decade. Figures below show frequency-response curves for 1/jω and jω, respectively. We can clearly see that the differences in the frequency responses of the factors 1/jω and jω lie in the signs of the slopes of the log magnitude curves and in the signs of the phase angles. Both log magnitudes become equal to 0 dB at ω=1.

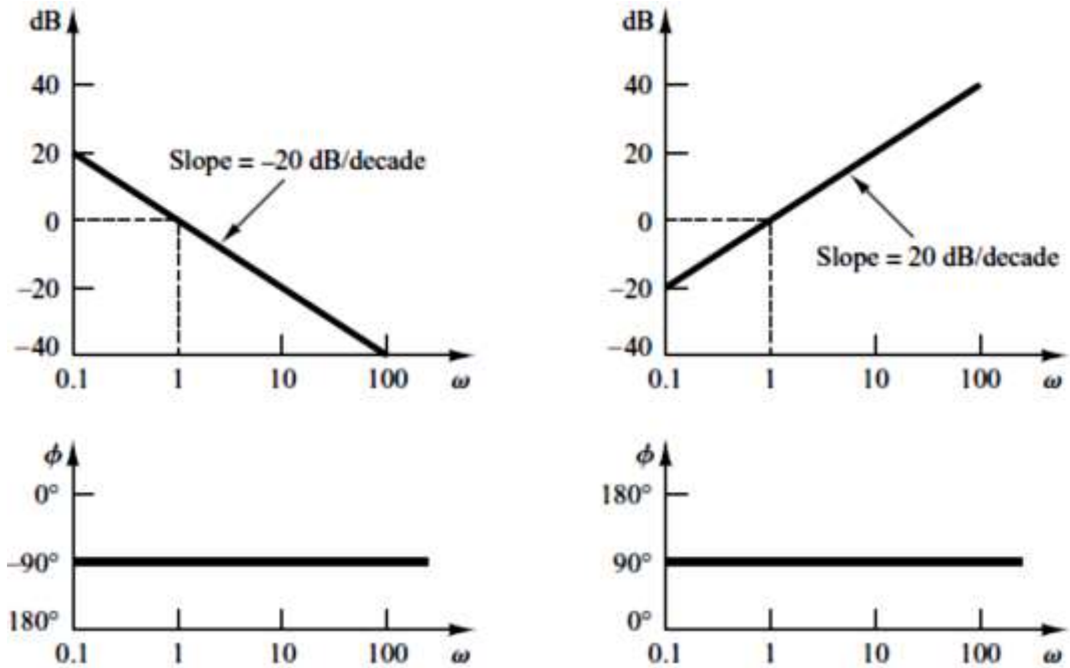
If the transfer function contains the factor $(1/j\omega)^n$ or $(j\omega)^n$, the log magnitude becomes, respectively,

$$20\log\left|\frac{1}{(j\omega)^n}\right| = -n \times 20\log|j\omega| = -20n\log\omega\ \text{dB}$$

or

$$20\log|(j\omega)^n| = n \times 20\log|j\omega| = 20n\log\omega\ \text{dB}$$

The slopes of the log-magnitude curves for the factors $(1/j\omega)^n$ and $(j\omega)^n$ are thus –20n dB/decade and 20n dB/decade, respectively. The phase angle of $(1/j\omega)^n$ is equal to **–90° x n** over the entire frequency range, while that of $(j\omega)^n$ is equal to **90° x n** over the entire frequency range. The magnitude curves will pass through the point (0 dB, ω=1).



Bode diagram of $G(j\omega) = 1/j\omega$

(a)

Bode diagram of $G(j\omega) = j\omega$

(b)

6.2.3 First-Order Factors $(1 + j\omega T)^{\pm 1}$

The log magnitude of the first-order factor 1/(1+jωT) is

$$20 \log \left| \frac{1}{1 + j\omega T} \right| = -20 \log \sqrt{1 + \omega^2 T^2} \text{ dB}$$

For low frequencies, such that ω « 1/T, the log magnitude may be approximated by

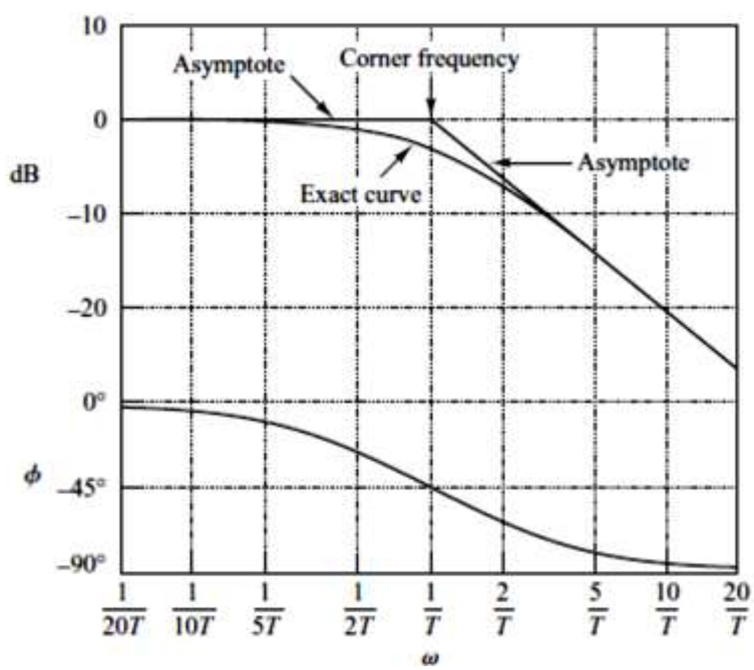
$$-20 \log \sqrt{1 + \omega^2 T^2} \doteq -20 \log 1 = 0 \text{ dB}$$

Thus, the log-magnitude curve at low frequencies is the constant 0-dB line. For high frequencies, such that $\omega \gg 1/T$,

$$-20 \log \sqrt{1 + \omega^2 T^2} \doteq -20 \log \omega T \text{ dB}$$

This is an approximate expression for the high-frequency range. At $\omega=1/T$, the log magnitude equals 0 dB; at $\omega=10/T$, the log magnitude is –20 dB. Thus, the value of –20 log ωT dB decreases by 20 dB for every decade of ω. For $\omega \gg 1/T$, the log-magnitude curve is thus a straight line with a slope of –20 dB/decade (or –6 dB/octave).

Our analysis shows that the logarithmic representation of the frequency-response curve of the factor $1/(1+j\omega T)$ can be approximated by two straight-line asymptotes, one a straight line at 0 dB for the frequency range $0< \omega <1/T$ and the other a straight line with slope –20 dB/decade (or –6 dB/octave) for the frequency range $1/T< \omega < \infty$. The exact log-magnitude curve, the asymptotes, and the exact phase-angle curve are shown in Figure below.



The frequency at which the two asymptotes meet is called the ***corner* frequency or *break* frequency**. For the factor 1/(1+jωT), the frequency ω=1/T is the corner frequency, since at ω=1/T the two asymptotes have the same value. (The low-frequency asymptotic expression at ω=1/T is 20 log 1 dB=0 dB, and the high-frequency asymptotic expression at ω=1/T is also 20 log 1 dB=0 dB.) The corner frequency divides the frequency-response curve into two regions: **a curve for the low-frequency region** and **a curve for the high-frequency region**. The corner frequency is very important in sketching logarithmic frequency-response curves. The exact phase angle f of the factor 1/(1+jωT) is

$$\phi = -\tan^{-1}\omega T$$

At zero frequency, the phase angle is 0°. At the corner frequency, the phase angle is

$$\phi = -\tan^{-1}\frac{T}{T} = -\tan^{-1}1 = -45^\circ$$

At infinity, the phase angle becomes –90°. Since the phase angle is given by an inverse tangent function, the phase angle is skew symmetric about the inflection point at f= –45°. The error in the magnitude curve caused by the use of asymptotes can be calculated. The maximum error occurs at the corner frequency and is approximately equal to –3 dB, since

$$-20\log\sqrt{1+1} + 20\log 1 = -10\log 2 = -3.03\text{ dB}$$

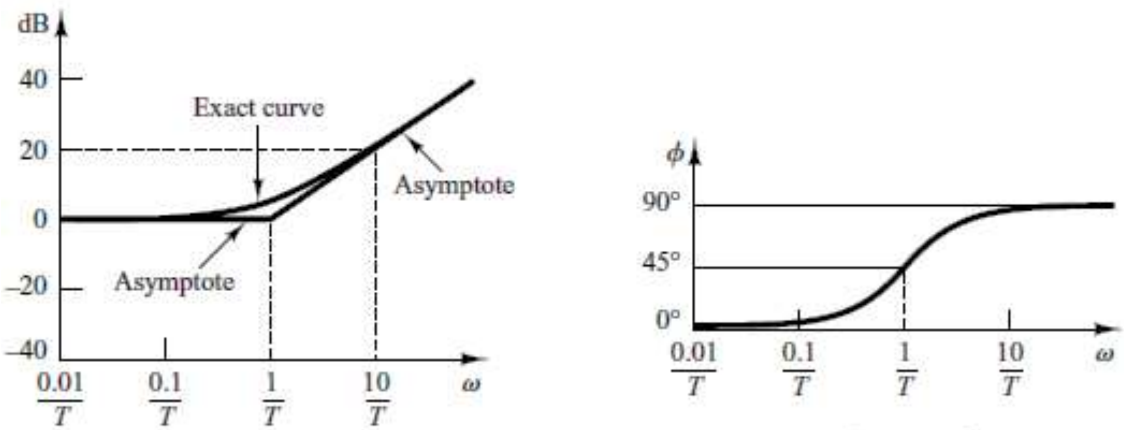
The error at the frequency one octave below the corner frequency, that is, at ω=1/(2T) is;

$$-20\log\sqrt{\frac{1}{4}+1} + 20\log 1 = -20\log\frac{\sqrt{5}}{2} = -0.97\text{ dB}$$

The error at the frequency one octave above the corner frequency, that is, at ω=2/T is

$$-20\log\sqrt{2^2+1} + 20\log 2 = -20\log\frac{\sqrt{5}}{2} = -0.97\text{ dB}$$

For the case where a given transfer function involves terms like $(1+j\omega T) < n$, a similar asymptotic construction may be made. The corner frequency is still at $\omega=1/T$, and the asymptotes are straight lines. The low-frequency asymptote is a horizontal straight line at 0 dB, while the high-frequency asymptote has the slope of –20n dB/decade or 20n dB/decade. The error involved in the asymptotic expressions is n times that for $(1+j\omega T)^{\pm 1} < 1$. The phase angle is *n* times that of $(1+j\omega T)^{\pm 1} < 1$ at each frequency point.



Log-magnitude curve, together with the asymptotes, and phase-angle curve for $1 + j\omega T$.

6.2.4 Quadratic Factors $[1 + 2\zeta(j\omega/\omega_n) + (j\omega/\omega_n)^2]^{\pm 1}$

Control systems often possess quadratic factors of the form

$$G(j\omega) = \frac{1}{1 + 2\zeta\left(j\frac{\omega}{\omega_n}\right) + \left(j\frac{\omega}{\omega_n}\right)^2}$$

If $\zeta > 1$, this quadratic factor can be expressed as a product of two first-order factors with real poles. If $0 < \zeta < 1$, this quadratic factor is the product of two complex conjugate factors. Asymptotic approximations to the frequency-response curves are not accurate for a factor with low values of ζ. This is because the magnitude and phase of the quadratic factor depend on both the corner frequency and the damping ratio ζ. The asymptotic frequency-response curve may be obtained as follows: Since

$$20\log\left|\frac{1}{1+2\zeta\left(j\frac{\omega}{\omega_n}\right)+\left(j\frac{\omega}{\omega_n}\right)^2}\right| = -20\log\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(2\zeta\frac{\omega}{\omega_n}\right)^2}$$

for low frequencies such that $\omega \ll \omega_n$, the log magnitude becomes

$$-20\log 1 = 0\ \text{dB}$$

The low-frequency asymptote is thus a horizontal line at 0 dB. For high frequencies such that $\omega \gg \omega_n$, the log magnitude becomes

$$-20\log\frac{\omega^2}{\omega_n^2} = -40\log\frac{\omega}{\omega_n}\ \text{dB}$$

The equation for the high-frequency asymptote is a straight line having the slope –40 dB/decade, since

$$-40\log\frac{10\omega}{\omega_n} = -40 - 40\log\frac{\omega}{\omega_n}$$

The high-frequency asymptote intersects the low-frequency one at $\omega=\omega_n$, since at this frequency

$$-40\log\frac{\omega_n}{\omega_n} = -40\log 1 = 0\ \text{dB}$$

This frequency, ω_n, is the corner frequency for the quadratic factor considered. The two asymptotes just derived are independent of the value of ζ. Near the frequency $\omega=\omega_n$, a resonant peak occurs. Figure 7–9 shows the exact log-magnitude curves, together with the straight-line asymptotes and the exact phase-angle curves for the quadratic factor with several values of ζ.

The phase angle of the quadratic factor $[1 + 2\zeta(j\omega/\omega_n) + (j\omega/\omega_n)^2]^{-1}$ is

$$\phi = \angle \frac{1}{1 + 2\zeta\left(j\frac{\omega}{\omega_n}\right) + \left(j\frac{\omega}{\omega_n}\right)^2} = -\tan^{-1}\left[\frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right]$$

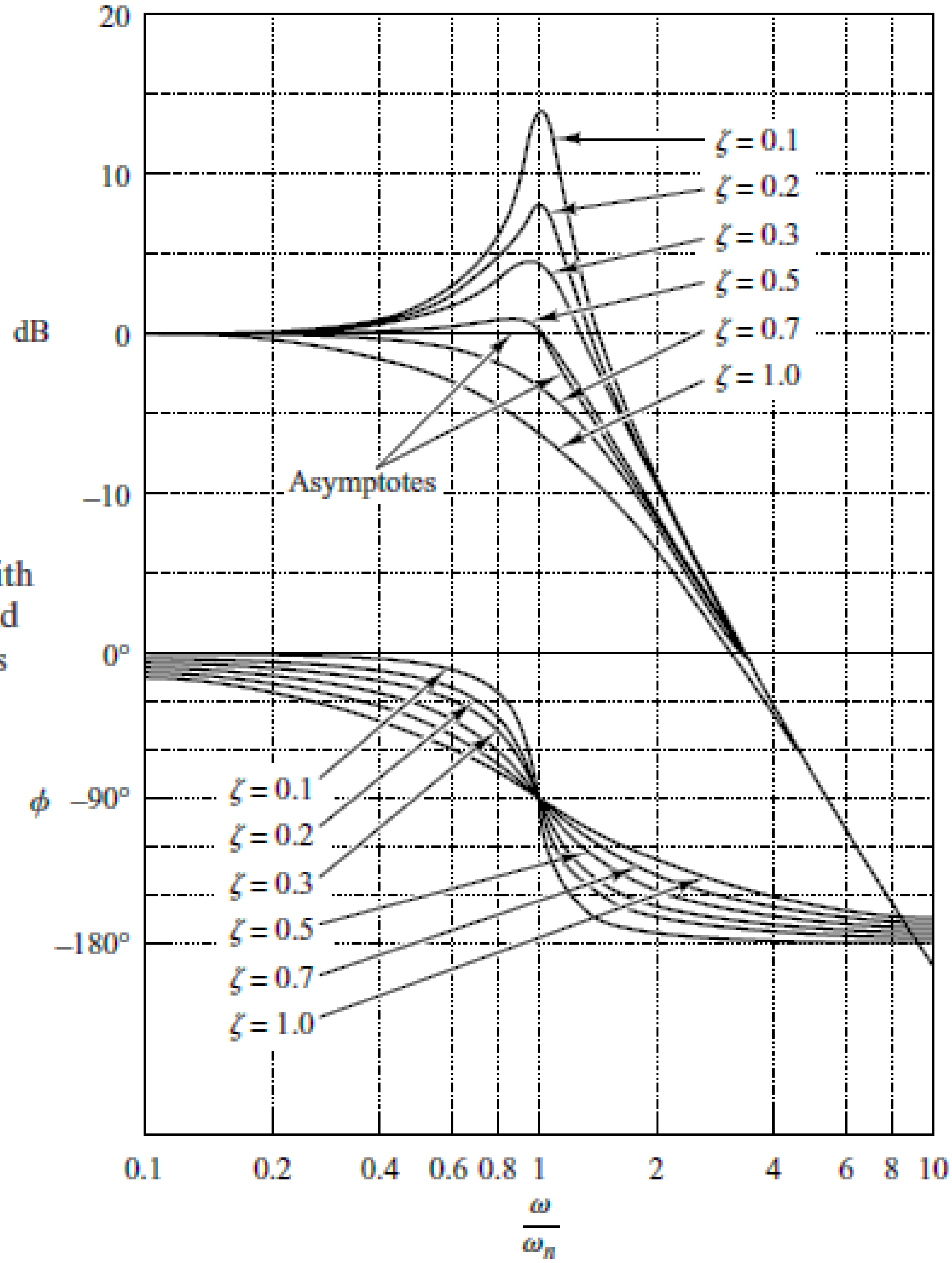


Figure 1

Log-magnitude curves, together with the asymptotes, and phase-angle curves of the quadratic transfer function

The phase angle is a function of both ω and ζ. At $\omega=0$, the phase angle equals 0°. At the corner frequency $\omega=\omega_n$, the phase angle is –90° regardless of ζ, since

$$\phi = -\tan^{-1}\left(\frac{2\zeta}{0}\right) = -\tan^{-1}\infty = -90°$$

At $\omega=\infty$, the phase angle becomes –180°. The phase-angle curve is skew symmetric about the inflection point—the point where $\phi=-90°$. There are no simple ways to sketch such phase curves. We need to refer to the phase-angle curves shown in Figure 1.

Example 6.1

Obtain the Bode plot of the system given by the transfer function

$$G(s) = \frac{1}{2s+1}$$

We convert the transfer function in the following format by substituting $s = j\omega$

$$G(j\omega) = \frac{1}{2j\omega + 1}$$

$$\omega << \frac{1}{2}$$ (low frequency)

i.e., for small values of ω

$G(j\omega) \approx 1.$

Therefore taking the log magnitude of the transfer function for very small values of ω, we get

$$20\log\left|G(j\omega)\right| = 20\log(1) = 0$$

Hence we see that below the break point the magnitude curve is approximately a constant. For,

$\omega >> \frac{1}{2}$ (High frequency)

i.e., for very large values of ω

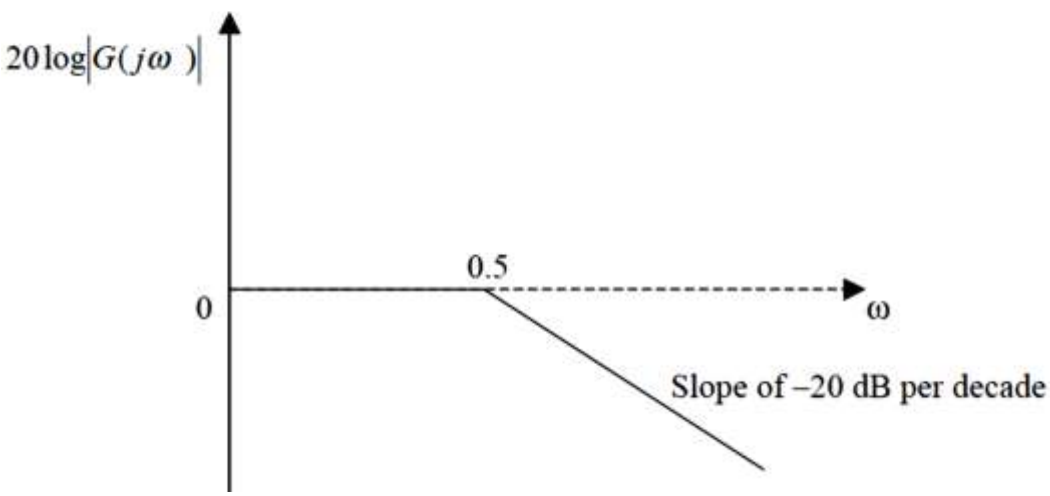
$$G(j\omega) \approx \frac{1}{2j\omega}$$

Similarly taking the log magnitude of the transfer function for very large values of ω, we have

$$20\log|G(j\omega)| = 20\log\left|\frac{1}{2j\omega}\right| = 20\log\left(\frac{1}{2\omega}\right)$$

$$= 20\log(1) - 20\log(2\omega) = -20\log(2\omega).$$

So we see that, above the break point the magnitude curve is linear in nature with a slope of –20 dB per decade. The two asymptotes meet at the break point. The asymptotic bode magnitude plot is shown below.



The phase of the transfer function is given by

$$\phi = 0 - \tan^{-1}(2\omega) = -\tan^{-1}(2\omega).$$

So for small values of ω i.e., $\omega \cong 0$, we get

$$\phi \approx 0.$$

For very large values of ω i.e., ω -> ∞, the phase tends to –90 degrees.

Example 6.2

Obtain the bode plot of the system given by the transfer function

$$G(s) = \frac{4}{s^2 + s + 4}$$

Substituting $s = j\omega$ in the above transfer function, we get

$$G(j\omega) = \frac{4}{(j\omega)^2 + j\omega + 4}$$

From the above transfer function, it can be concluded that $\omega_n = 2$, so therefore reducing the above transfer function by dividing both the numerator and denominator by ω_n, we get

$$G(j\omega) = \frac{1}{\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\left(\frac{j\omega}{\omega_n}\right) + 1} = \frac{1}{\left(\frac{j\omega}{2}\right)^2 + 0.5\left(\frac{j\omega}{2}\right) + 1}$$

In this case the break point is $\omega = \omega_n$. Therefore for

$\omega << \omega_n$, i.e., for small values of ω,

$$G(j\omega) \approx 1.$$

Taking the log magnitude, we get

$$20\log|G(j\omega)| = 20\log(1) = 0.$$

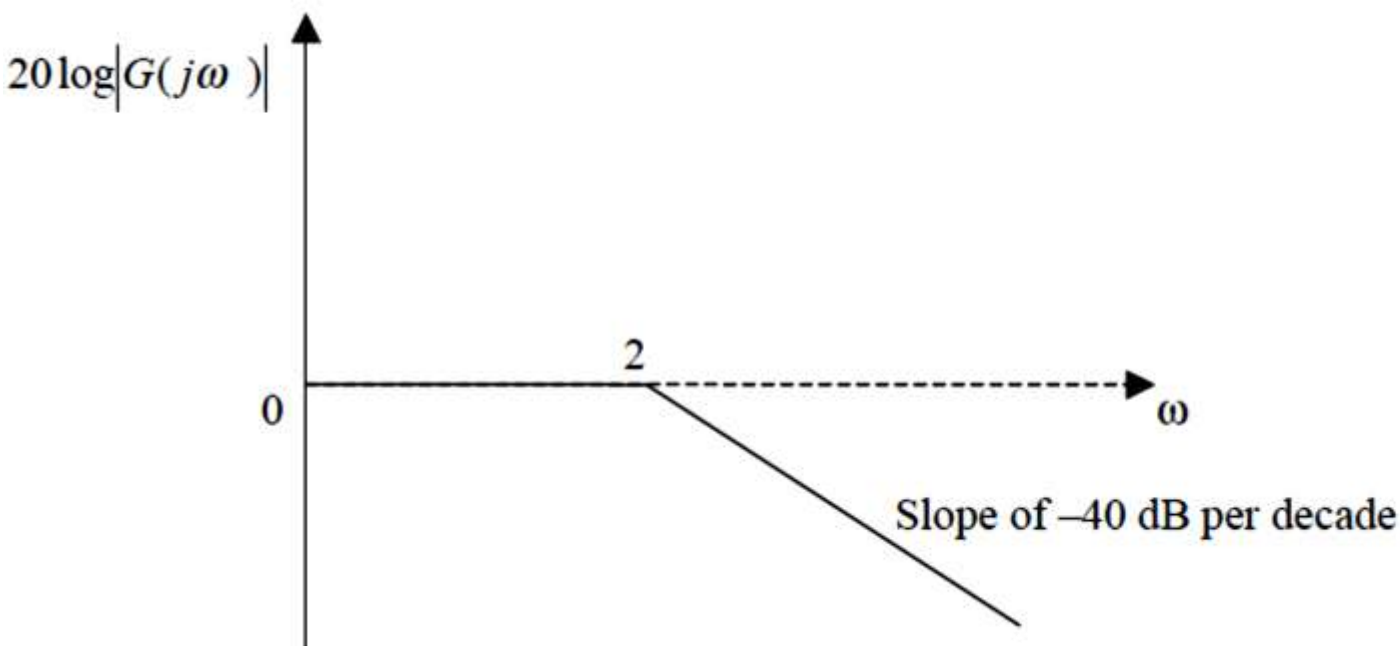
Therefore the magnitude is approximately a constant below the break point. For larger values of ω i.e., for $n\ \omega >> \omega_n$, we get

$$G(j\omega) \approx \frac{1}{\left(\frac{j\omega}{\omega_n}\right)^2} = \frac{1}{\left(\frac{j\omega}{2}\right)^2}$$

Taking the log magnitude, we get

$$20\log|G(j\omega)| = 20\log\left(\frac{1}{\left(\frac{\omega}{\omega_n}\right)^2}\right) = 20\log(1) - 20\log\left(\frac{\omega}{\omega_n}\right)^2 = -40\log\frac{\omega}{\omega_n} = -40\log\frac{\omega}{2}$$

From the above relation, it can be concluded that the magnitude plot is linear in nature with a slope of –40 dB per decade. The asymptotic plot is as shown below.



The transfer function can be rewritten as

$G(s) = \frac{1}{(s+a)(s+b)}$, where 'a' and 'b' are the roots of the denominator.

Substituting $s = j\omega$, we get

$$G(j\omega) = \frac{1}{(j\omega + a)(j\omega + b)}.$$

The phase of the above transfer function is given as

$$\phi = 0 - \tan^{-1}(\frac{\omega}{a}) - \tan^{-1}(\frac{\omega}{b}).$$

So therefore for $\omega \cong 0$, we get

$$\phi \approx 0.$$

For very large values of ω, i.e., $\omega \to \infty$, the phase tends to –180 degrees.

Example 6.3

Plot the Bode magnitude and phase for the system with transfer function

$$G(s) = \frac{2000(s+0.5)}{s(s+10)(s+50)}.$$

Step 1: We convert the function to the form given below by substituting $s = j\omega$

$$G(j\omega) = \frac{2000(j\omega + 0.5)}{j\omega\,(j\omega + 10)(j\omega + 50)} = \frac{2\left(\frac{j\omega}{0.5} + 1\right)}{j\omega\left(\frac{j\omega}{10} + 1\right)\left(\frac{j\omega}{50} + 1\right)}.$$

Step 2: We note that the term in jω is first order and in the denominator, so n = -1.

Therefore, the low frequency asymptote is defined by the first term:

$$G(j\omega) = \frac{2}{j\omega}.$$

This asymptote is valid for $\omega < 0.1$ because the lowest break point is at $\omega = 0.5$. The magnitude plot of this term has a slope of –1 or –20 dB per decade. We locate the magnitude by passing through the value 2 at $\omega = 1$ even though the composite curve will not go through this point because of the break point at $\omega = 0.5$.

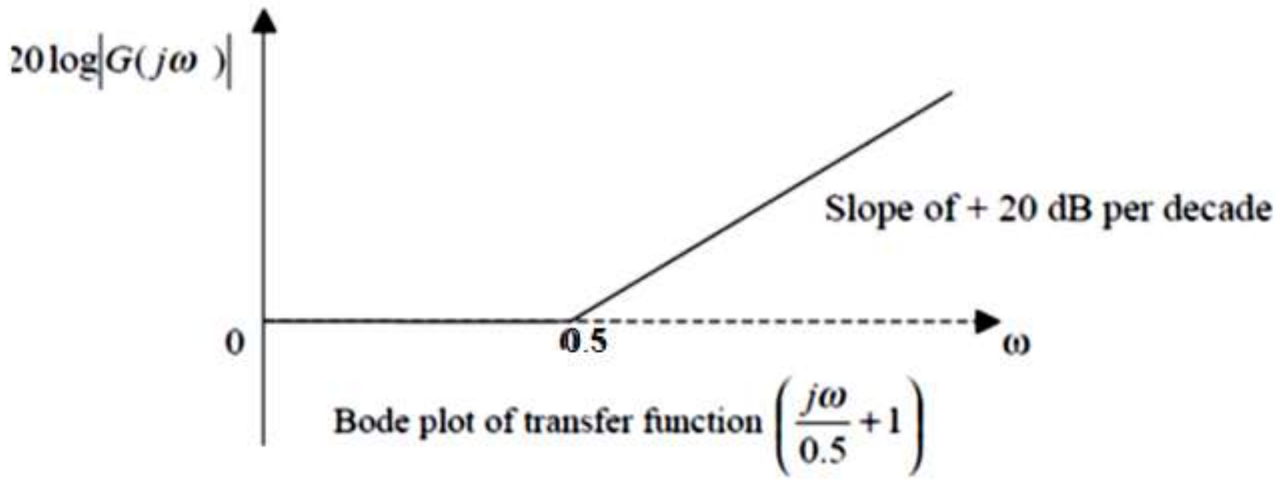
Step 3: We obtain the remainder of asymptotes as shown in the figure. First we draw a line with 0 slope that intersects the original –1 slope at $\omega = 0.5$. Then we draw a –1 slope line that intersects the previous one at $\omega = 10$. Finally, we draw a –2 slope line that intersects the previous –1 slope at $\omega = 50$.

Step 4: We then sketch the actual curve by calculating the value of the magnitude at the break points and joining those points by a smooth curve. We see that the actual curve is approximately tangential to the asymptotes when far away from the break points and are a factor of 1.4 (+ 3 dB) above the asymptote at $\omega = 0.5$ break point and a factor of 0.7 (-3 dB) below the asymptote at $\omega = 10$ and $\omega = 50$ break points.

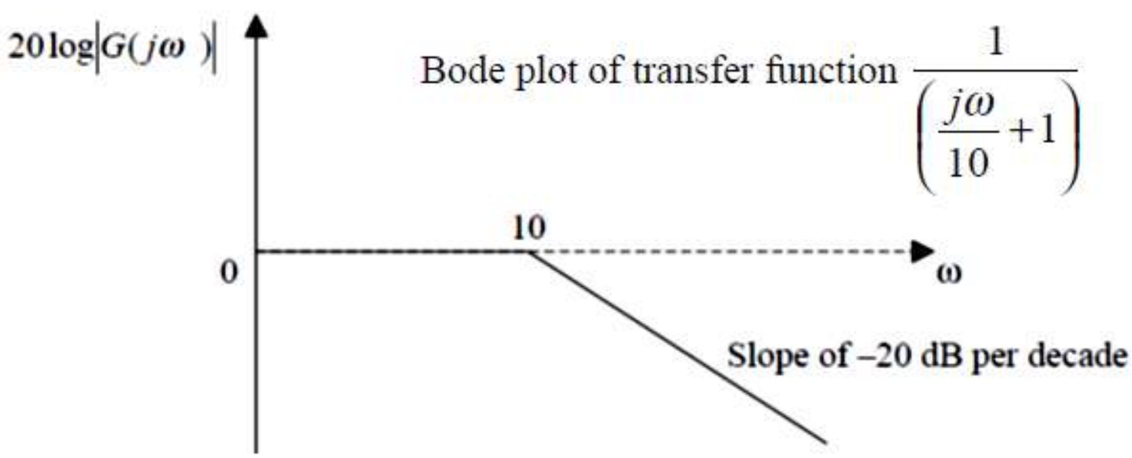
Step 5: Since the phase of is $(2/j\omega)$ -90°, the phase curve starts at -90° at the lowest frequencies.

Step 6: The individual phase curves are shown in the form of dashed line. Note that the composite curve approaches each individual term.

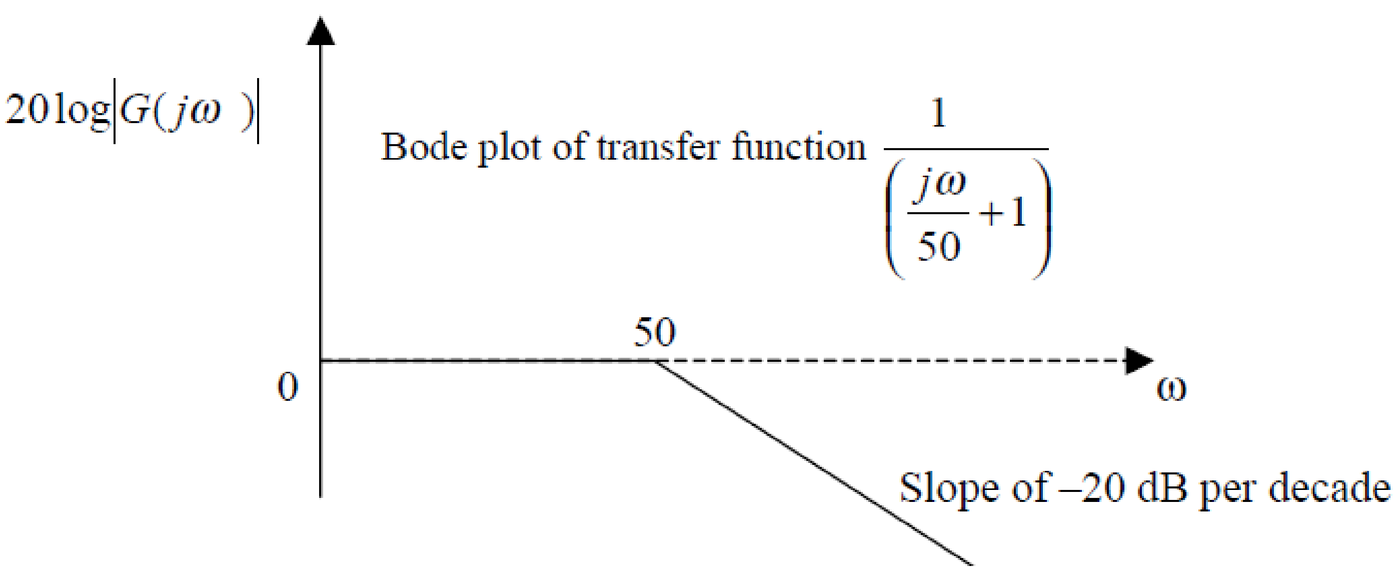
The following plots depict the bode magnitude plot of the individual terms in the transfer function.



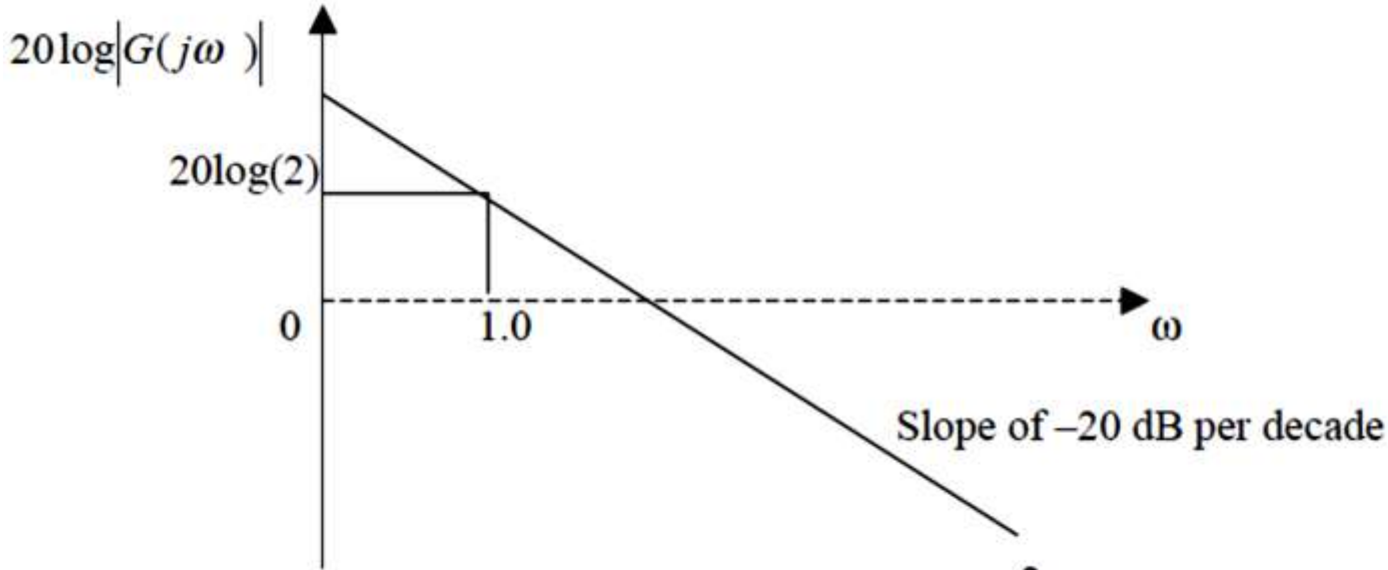
20log|G(jω)|
Slope of + 20 dB per decade
0
0.5
ω
Bode plot of transfer function (jω/0.5 + 1)



20log|G(jω)|
Bode plot of transfer function 1/(jω/10 + 1)
10
0
ω
Slope of –20 dB per decade

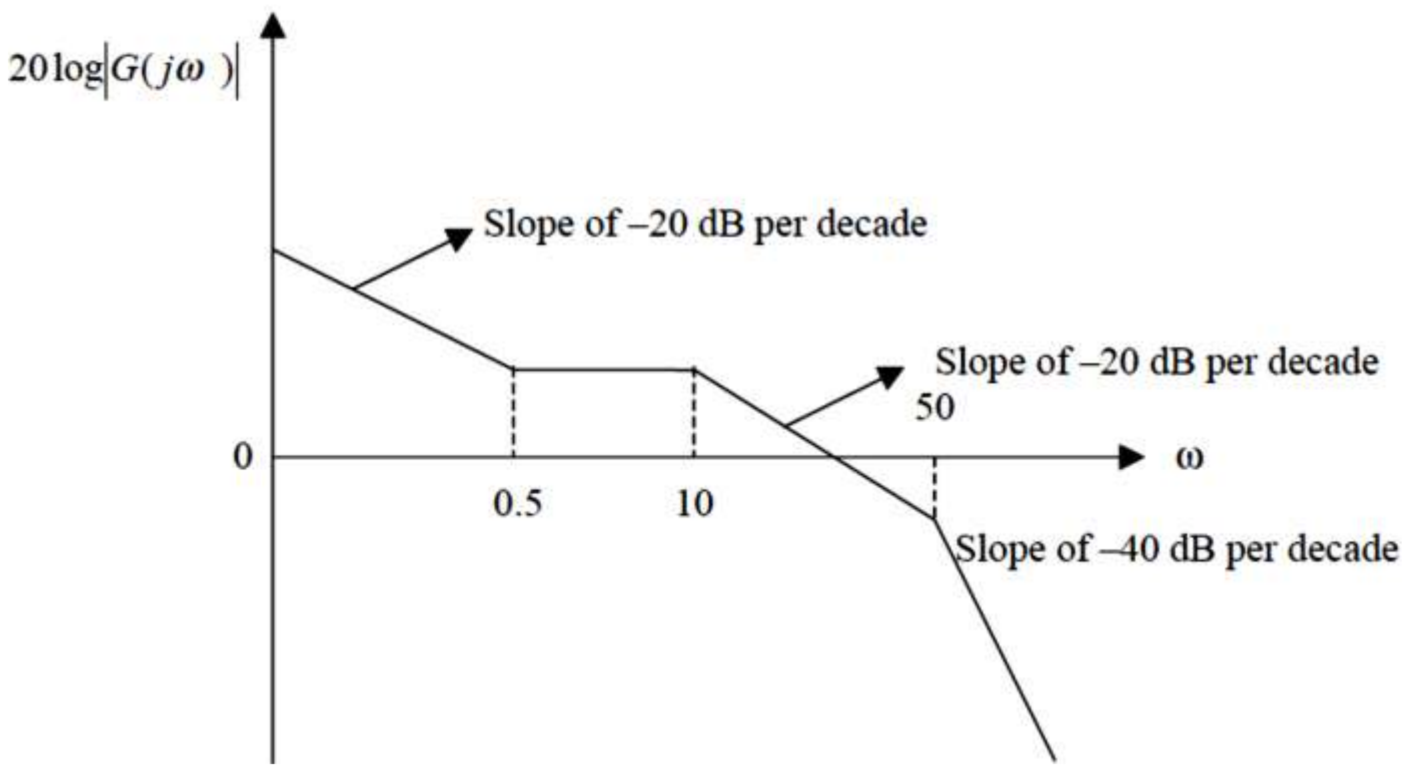


20log|G(jω)|
Bode plot of transfer function 1/(jω/50 + 1)
50
0
ω
Slope of –20 dB per decade



Bode plot of transfer function $\frac{2}{j\omega}$

Combining the above bode diagrams, the composite asymptotic curve is as shown below.



Example 6.4

Draw the frequency response of the system given by the transfer function

$$G(s) = \frac{10}{s(s^2 + 0.4s + 4)}.$$

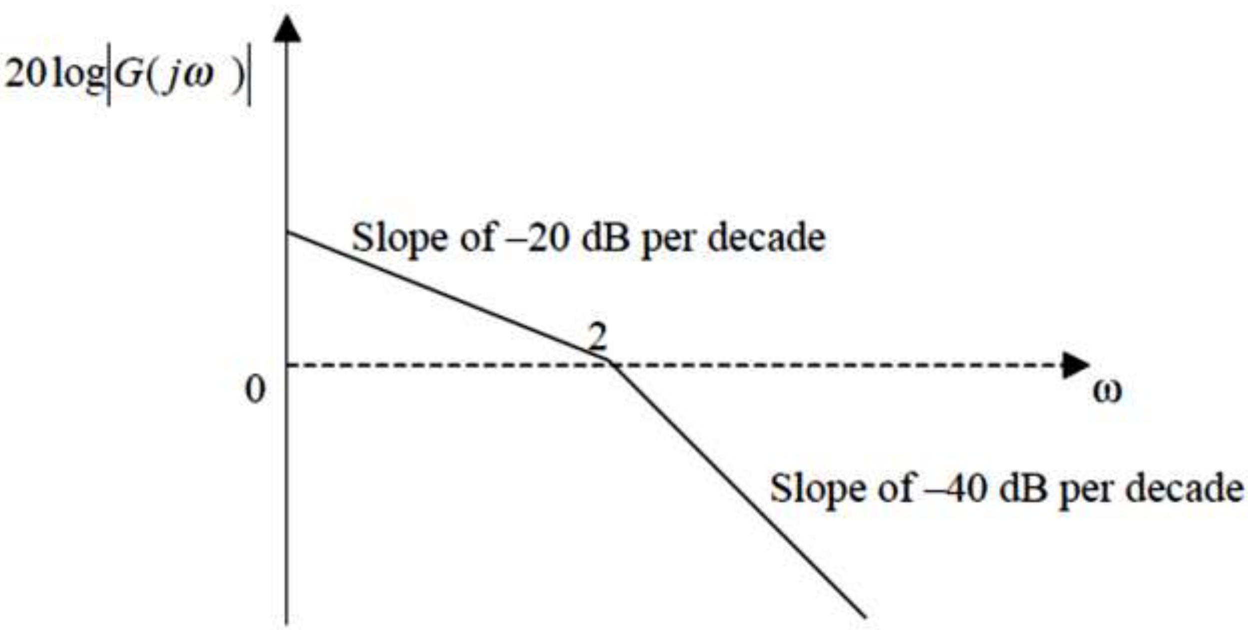
Rewrite the above transfer function as

$$G(s) = \frac{10}{4} \frac{1}{s\left(\frac{s^2}{4} + 0.2\frac{s}{2} + 1\right)}.$$

Substituting $s = j\omega$ in the above transfer function, we get

$$G(j\omega) = \frac{10}{4} \frac{1}{j\omega\left(\left(\frac{j\omega}{2}\right)^2 + 0.2\frac{j\omega}{2} + 1\right)}.$$

The breakpoint for the above transfer function is at ω = 2. Following the same procedure as in example 3, the composite asymptotic bode magnitude curve is as shown below.



Assignments:

Draw the Bode plot for each of the following systems.

$$a) \quad G(s) = \frac{4000}{s(s+40)},$$

$$b) \quad G(s) = \frac{1}{(s+1)(s^2+s+4)},$$

$$c) \quad G(s) = \frac{10(s+4)}{s(s+1)(s^2+25+5)}.$$