Introduction to Digital Control System

1.1 Why Digital Control?

Digital control offers distinct advantages over analog control that explain its popularity. Here are some of its many advantages:

a- Accuracy. Digital signals are represented in terms of zeros and ones with typically 12 bits or more to represent a single number.

b- Implementation errors. Digital processing of control signals involves addition and multiplication by stored numerical values. The errors that result from digital representation and arithmetic are negligible.

c- Flexibility. An analog controller is difficult to modify or redesign once implemented in hardware. A digital controller is implemented in firmware or software, and its modification is possible without a complete replacement of the original controller.

d -Speed. The speed of computer hardware has increased exponentially since the 1980s. This increase in processing speed has made it possible to sample and process control signals at very high speeds. Because the interval between samples, the sampling period, can be made very small, digital controllers achieve performance that is essentially the same as that based on continuous monitoring of the controlled variable.

e -Cost. Although the prices of most goods and services have steadily increased, the cost of digital circuitry continues to decrease. Advances in very large scale integration (VLSI) technology have made it possible to manufacture better, faster, and more reliable integrated circuits and to offer them to the consumer at a lower price. This has made the use of digital controllers more economical even for small, low-cost applications.

1.2. Building a Closed Loop Digital control system and its components

In recent years, Microprocessors and Microcomputers are used in the control systems to obtain necessary controlling action. Such controllers use digital signals which exists only at finite instants, in the form of short pulses (Digital controllers). Thus the digital control system is hybrid system using the combination of continuous time signals & digital signals. To obtain analog signal from digital, Digital to Analog (**DAC**) converters are used while to obtain digital signals from analog, Analog to Digital (**ADC**) converters are used. The input & output of a digital controller are both in digital form. The digital signals exist in the form of coded digital data at discrete intervals of time. Such signals are obtained from computers, microprocessors, ADC and digital sensing elements. A typical digital control system is shown in Figure (1.1).

Applications: radar tracking systems, Industrial robots, Modern Industrial control system, aircraft control systems.

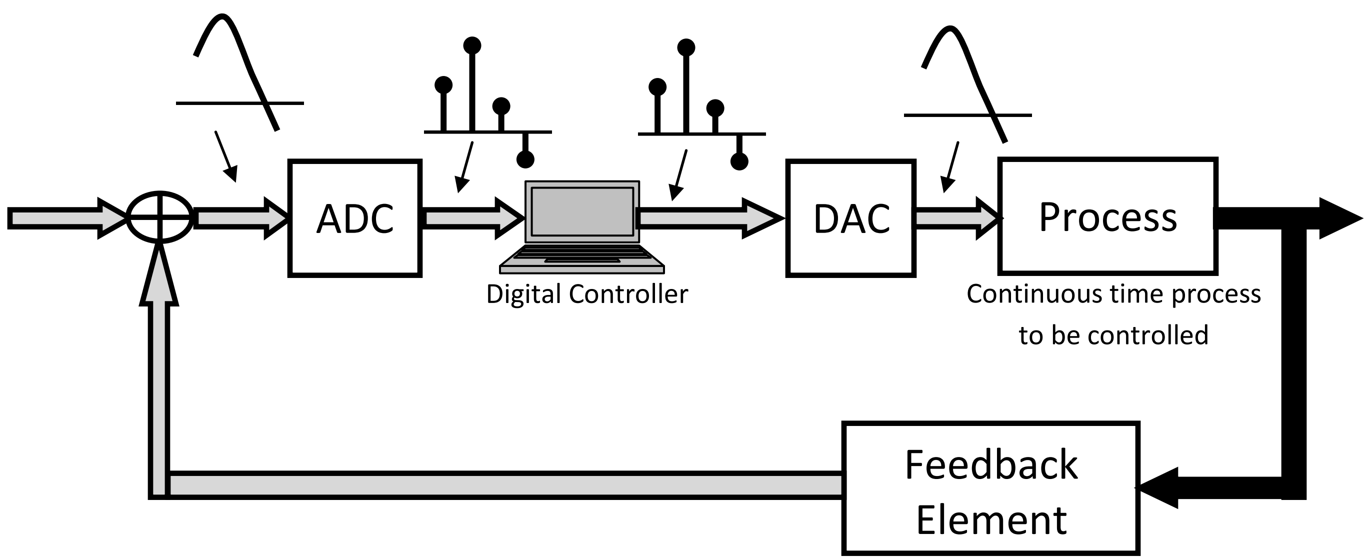


Figure (1.1) Typical Digital Control System

Many times digital signals are obtained by sampling the continuous time signals at regular intervals (T). The **switch** can be used as a **sampler**. When switch is closed for short duration of time, signal is available at the output and otherwise it is zero. Such signal is called sampled signal which exists in a digital form. The digital controllers accept such sampled error signals to produce controlled variable in digital form. This is converted to analog signal using DAC and hold circuits. The hold circuits convert sampled signal back to analog signal. This signal is used to control the process. The system using such sampler and hold circuits is called sampled data control system. The input and feedback signals both are continuous in nature. The accuracy of sampled signals is less than the digital signals. Hence digital control systems are more accurate than sampled data systems.

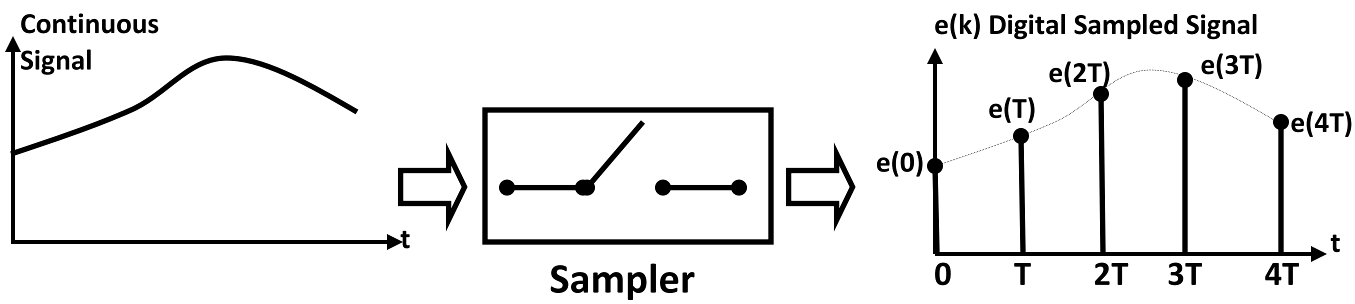


Figure (1.2) Typical Sampling method

The digital sampled signal may be described mathematically as;
$e(k) = e(0) + e(T) + e(2T) + e(4T)$

1.3. Z-Transform:

A transform will now be defined that can be utilized in the analysis of discrete time systems modeled by difference equations. The function E(Z) is defined as a power series in the Z^{-k} with coefficients equal to the values of number sequence {e(k)}. This transform, called the **z-transform**, is then expressed by the transform below:

$$\boldsymbol{E(z) = \text{Ƶ}\{e(kT)\} = e(0) + e(T)z^{-T} + e(2)z^{-2T} + e(3)z^{-3T}\ldots\ldots\ldots.}$$

Where Ƶ(*) indicates the z-transform operation and Z^{-1}(*) indicates the inverse z-transform. E(z) may also be written in more compact notation as:

$$E(z) = \mathcal{Z}\{e(kT)\} = \sum_{k=0}^{\infty} e(kT)z^{-kT}$$

And for T=1;

$$E(z) = \mathcal{Z}\{e(kT)\} = \sum_{k=0}^{\infty} e(k)z^{-k}$$

And;

$$E(Z) = e(0) + e(1)Z^{-1} + e(2)Z^{-2} + e(3)Z^{-3} + e(4)Z^{-4} + e(5)Z^{-5} \ldots\ldots\ldots\ldots$$

It should be remembered that z-transform applies to a sequence.

Example 1.1:

Given E(z) below, find e(k).

$$E(z) = 1 + 3z^{-1} - 2z^{-2} + z^{-4} + \cdots$$

Solution:

Compare this equation with the z-transform definition, then

$$e(0) = 1 \qquad e(3) = 0$$
$$e(1) = 3 \qquad e(4) = 1$$
$$e(2) = -2 \qquad \cdots$$

--End of Example-----------

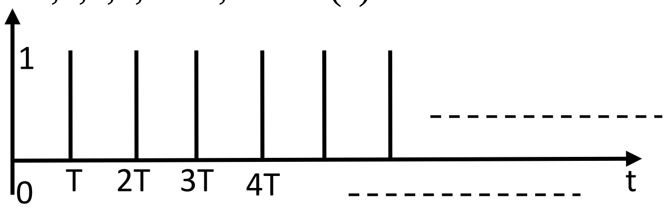
Now consider the below Identity :

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots, \qquad |x| < 1$$

Which is used to convert from **open loop** to **closed loop** form.

Example 1.2:

Given that e(k) = 1 for all values of k=0,1,2,3, ..., find E(z).



Solution:

Comparing with definition equation,

$$E(z) = 1 + z^{-1} + z^{-2} + \cdots$$

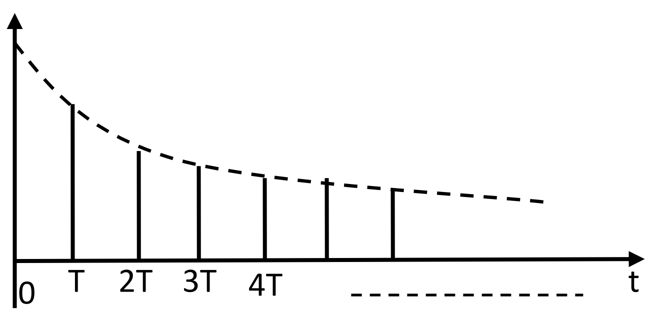
And converting to closed form loop;

$$E(z) = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}, \qquad |z^{-1}| < 1$$

---End of Example-----------

Example 1.3:

For the following sampled functions, find E(z) using Z.T. definition.



1- $e(k) = e^{-akT}\, u(k)$

Solutions:

Substitute in definition equation for k=0,1,2,3, ...

$E(z) = 1 + e^{-aT} z^{-1} + e^{-a2T} z^{-2} + e^{-a3T} z^{-3} + \ldots\ldots$

$E(z) = 1 + (e^{-aT} z^{-1}) + (e^{-aT} z^{-1})^2 + (e^{-aT} z^{-1})^3 + \ldots\ldots$

Now E(z) may be converted to a closed form by applying (2-10), as below;

$$E(z) = \frac{1}{1 - \epsilon^{-aT} z^{-1}} = \frac{z}{z - \epsilon^{-aT}}, \qquad |\epsilon^{-aT} z^{-1}| < 1$$

2- e(k) = 4^k u(k)

$$E(Z) = 1 + 4\,Z^{-1} + 4^2\,Z^{-2} + 4^3\,Z^{-3} + 4^4\,Z^{-4} + \ldots\ldots..$$

$$E(Z) = \frac{1}{1 - 4\,Z^{-1}} = \frac{Z}{Z-4}$$

2- e(k) = 2^{-1-k} u(k)

Simplifying e(k)

$$e(k) = 2^{-1}(2^{-1})^k\,u(k)$$

$$E(Z) = 2^{-1}[\,1 + (2^{-1})\quad Z^{-1} + (2^{-1})^2\,Z^{-2} + (2^{-1})^3\,Z^{-3} + (2^{-1})^4\,Z^{-4} + \ldots\ldots..$$

then;

$$E(Z) = \frac{2^{-1}}{1 - (2^{-1})\quad Z^{-1}} = \frac{0.5Z}{Z - 0.5}$$

3- $\boldsymbol{e(k) = \delta(k)}$

$$E(Z) = \delta(0) + \delta(1)\,Z^{-1} + \delta(2)\,Z^{-2} + \delta(3)\,Z^{-3} + \delta(4)\,Z^{-4} + \ldots\ldots..$$

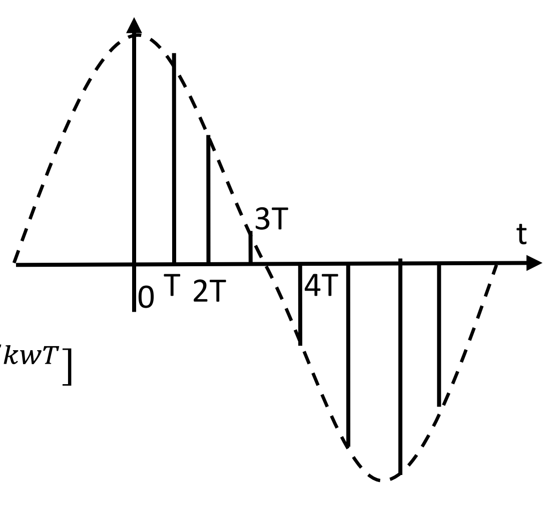
And since $\delta(k)$ is zero everywhere except for $\delta(0) = 1$, then;

$$E(Z) = 1$$

<u>Exercise:</u>

Consider the sampled cosine function :

$$f_k = f(kT) = \begin{Bmatrix} 0 & k < 0 \\ \cos kwT & k \geq 0 \end{Bmatrix}$$



Use Euler identity: $\cos kwT = \frac{1}{2}[e^{jkwT} + e^{-jkwT}]$

Find F(z).

1.4 Properties Of The z-Transform:

1- Addition and Subtraction.

Property. The z-transform of a sum of number sequence is equal to the sum of the z-transform of the number sequences; that is;

$$\mathcal{Z}[e_1(k) \pm e_2(k)] = E_1(z) \pm E_2(z)$$

Proof. From the definition of the z-transform,

$$\mathcal{Z}[e_1(k) \pm e_2(k)] = \sum_{k=0}^{\infty} [e_1(k) \pm e_2(k)]z^{-k}$$

$$= \sum_{k=0}^{\infty} e_1(k)z^{-k} \pm \sum_{k=0}^{\infty} e_2(k)z^{-k} = E_1(z) \pm E_2(z)$$

2- Multiplication by a constant.

Property. The z-transform of a number sequence multiplied by a constant is equal to the constant multiplied by the z-transform of the number sequence:

$$\mathcal{Z}[ae(k)] = a\mathcal{Z}[e(k)] = aE(z)$$

Proof. From the definition;

$$\mathcal{Z}[ae(k)] = \sum_{k=0}^{\infty} ae(k)z^{-k} = a\sum_{k=0}^{\infty} e(k)z^{-k} = aE(z)$$

The linearity property of the z-transform can also be proved as follows:

$$e(k) = ae_1(k) + be_2(k)$$

Then;

$$\mathcal{Z}[e(k)] = \mathcal{Z}[ae_1(k) + be_2(k)]$$

$$= \sum_{k=0}^{\infty} [ae_1(k) + be_2(k)]z^{-k}$$

$$= a \sum_{k=0}^{\infty} e_1(k) z^{-k} + b \sum_{k=0}^{\infty} e_2(k) z^{-k}$$

$$= a \mathcal{Z}[e_1(k)] + b \mathcal{Z}[e_2(k)]$$

$$= aE_1(z) + bE_2(z)$$

3- Real translation (shifting Left and Right).

Property. Let n be a positive integer, then;

$$\mathcal{Z}\{e(k-n)\,u(k-n)\} = z^{-n}\,E(z)$$

and

$$\mathcal{Z}\{e(k+n)u(k+n)\} = z^{n}\left[E(z) - \sum_{k=0}^{n-1} e(k)\,z^{-k}\right]$$

Using the unit step function, the z-T of e(k) may be written as;

$$\mathcal{Z}\{e(k)\} = \mathcal{Z}\{e(k)\,u(k)\}$$

Where u(k) is the discrete unit step function defined by;

$$u(k) = \begin{cases} 0, & k < 0 \\ 1, & k \ge 0 \end{cases}$$

The time-delayed function e(k-n) may be given by;

$$e(k-n)u(k-n) = e(k)u(k)\big|_{k \leftarrow k-n}$$

advancing e(k) in time results in

$$\mathfrak{z}[e(k)u(k)|_{k \leftarrow k+n}] = \mathfrak{z}[e(k+n)u(k+n)]$$
$$= \mathfrak{z}[e(k+n)u(k)]$$

To illustrate these properties further, consider the number sequence shown in Table 1.1, which illustrates the effects of shifting by two sample periods. For the sequence e(k-2) u(k-2), no numbers of sequence e(k) are lost; thus the z-transform of e(k-2) u(k-2) can be expressed as a simple function of E(z). However, in forming the sequence e(k+2)u(k), the first two values of e(k) have been lost, and the z-transform of e(k+2)u(k) cannot be expressed as a simple function of E(z).

Table 1.1 Example showing shifting property

k	e(k)	e(k-2)	e(k+2)
0	2	0	1.3
1	1.6	0	1.1
2	1.3	2	1
3	1.1	1.6	…….
4	1	1.3	…….
……	…….	…….	…….

Example 1.4:

It was shown in Example 1.3 that;

$$\mathfrak{z}[\epsilon^{-akT}] = \frac{z}{z - \epsilon^{-aT}}$$

Applying the shifting property;

$$\mathfrak{z}[\epsilon^{-a(k-3)T} u[(k-3)T]] = z^{-3}\left[\frac{z}{z - \epsilon^{-aT}}\right] = \frac{1}{z^2(z - \epsilon^{-aT})}$$

Where u(kT) is a unit step. Also;

$$\mathcal{Z}[\epsilon^{-a(k+2)T}u(kT)] = z^2\left[\frac{z}{z-\epsilon^{-aT}} - 1 - \epsilon^{-aT}z^{-1}\right]$$

---End of Example-----------

We now define the discrete unit impulse fuction $\delta(k-n)$:

$$\delta(k-n) = \begin{cases} 1, & k = n \\ 0, & k \neq n \end{cases} \qquad (2\text{-}16)$$

This function is also called the unit sample fuction. The z-transform of the unit impulse function is then, for $n \geq 0$, ;

$$\mathcal{Z}[\delta(k-n)] = \sum_{k=0}^{\infty} \delta(k-n)z^{-k} = \delta(k-n)z^{-n} = z^{-n}$$

Example 1.5:

Find the Z.T. for the following discrete functions:

1- $\boldsymbol{e(k) = \delta(k-1)u(k-1)}$

$E(Z) = Z^{-1}$

2- $\boldsymbol{e(k) = 2^{2-k}\ u(k-1)}$

$\mathrm{e(k) = 4 * (2^{-1})^k\ u(k-1)} \rightarrow \quad \mathrm{E(z)} = \dfrac{4}{0.5Z-1}$

3- $\boldsymbol{e(k) = k\ u(k-1)}$

$\mathrm{e(k) = (k-1+1)\ u(k-1)} = (k-1)u(k-1) + u(k-1)$

then

$$\text{E(z)} = z^{-1}\frac{z}{(z-1)^2} + z^{-1}\frac{z}{(z-1)} = \frac{1+z-1}{(z-1)^2} = \frac{z}{(z-1)^2}$$

4- $\boldsymbol{e(k) = k^2\ u(k-2)}$

$\text{e(k)} = (\text{k}-2+2)^2\ \text{u(k}-2) = (\text{k}-2)^2\text{u(k}-2) + 4(k-2)\text{u(k}-2) + 4\text{u(k}-2)$

then;

$$\text{E(z)} = z^{-2}\frac{z(z+1)}{(z-1)^3} + 4\,z^{-2}\frac{z}{(z-1)^2} + 4\,z^{-2}\frac{z}{(z-1)}$$

5- $\boldsymbol{e(k) = sin(3k-9)\ u(k-3)}$

$\text{e(k)} = \text{sin3(k}-3)\ \text{u(k}-3)$

$$\text{E(z)} = z^{-3}\frac{z\,sin3}{z^2 - 2\,z\,cos3 + 1}$$

6- $\boldsymbol{e(k) = e^{-k}\,(k-1)\,u(k-1)}$

$$E(z) = \left[z^{-1}\frac{z}{(z-1)^2}\right]_{z=\frac{z}{e^{-1}}=ez} = \frac{1}{(ez-1)^2}$$

7- $\boldsymbol{e(k) = (k+1)\,u(k+1)}$

$$E(z) = Z^1\left[\frac{z}{(z-1)^2} - \sum_{k=0}^{0} k\,u(k)z^{-k}\right] = \frac{z^2}{(z-1)^2}$$

8- $\boldsymbol{e(k) = (k+2)\,u(k+2)}$

$$E(z) = Z^2\left[\frac{z}{(z-1)^2} - \sum_{k=0}^{1} k\,u(k)z^{-k}\right] = \frac{z^3}{(z-1)^2} - Z^2(Z^{-1}) = \frac{z^3}{(z-1)^2} - z$$

9- $\boldsymbol{e(k) = \delta(k+1)u(k+1)}$ **=0**

10- $\boldsymbol{e(k) = sin(2k+4)u(k+2)}$

$\text{e(k)} = \text{sin2(k}+2)\text{u(k}+2)$

$$E(z) = Z^2\left[\frac{z\,sin2}{z^2-2zcos2+1} - \sum_{k=0}^{1} sin2k\,u(k)z^{-k}\right]$$

$$E(z) = \frac{Z^3 sin2}{z^2-2zcos2+1} - \ z\,sin2$$

Exercises:

For the following discrete functions, find the Z.T.:

1- $e(k) = 3^k\ sin(6k-9)u(k-3)$

2- If f(k)=2^k cos3k u(k) then find the Z-transform of
[f(k+2) u(k) + f(k-3) u(k-3) + 2^k f(k) u(k)].

3- Find the z-Transform of $2^k\ [a(k-1)u(k-1)] - \ e^{2k}[a(k+2)u(k+2)]$
If $a(k) = 2k\ u(k)$

4- Find the z-Transform of $[b(k-2)u(k-2)] - \ 4^{k-1}[b(k+2)u(k+2)]$
If $b(k) = \sin\frac{\pi}{2}k\ \ u(k)$

5- Find the z-Transform of $4\delta(k+1) + 10\delta(k-2) + 2\sin\frac{\pi}{4}(k+2)u(k+2)$

6- Find the z-Transform of $6k^2 + 2^k\delta(k-4) + 2\cos\frac{\pi}{4}(k+1)u(k+1)$

7- Find the z-Transform of $3^k\ [e(k+2)u(k)] - \ e^{-k}[e(k-3)u(k-3)]$
If $e(k) = k\ e^{-k}u(k)$

8- Find the z-Transform of $4^k[c(k-1)u(k-1)] - \ 4^k[c(k+1)u(k+1)]$
If $c(k) = \cos\frac{\pi}{2}k\ \ u(k)$

9- Find the z-Transform of $\delta(k+2) + 2\delta(k-1) + 2\cos\frac{\pi}{4}(k+2)u(k+2)$

10- Find the z-Transform of $2^k k + \delta(k-1) + 2\sin\frac{\pi}{2}(k+1)u(k+1)$

4- Complex Translation:

Property. Given that z-transform of e(k) is E(z). then;

$$\mathcal{Z}\{e^{ak}\, e(k)\} = E(z)_{at\ z=\frac{z}{e^{a}}} = E(z)_{at\ z=ze^{-a}} = E(ze^{-a})$$

Proof. From the definition of the z-transform,

$$\mathfrak{z}[\epsilon^{ak} e(k)] = e(0) + \epsilon^{a} e(1)z^{-1} + \epsilon^{2a} e(2)z^{-2} + \cdots$$
$$= e(0) + e(1)(z\epsilon^{-a})^{-1} + e(2)(z\epsilon^{-a})^{-2} + \cdots$$

or

$$\mathfrak{z}[\epsilon^{ak} e(k)] = E(z)|_{z \leftarrow z\epsilon^{-a}} = E(z\epsilon^{-a})$$

The property may be written as;

$$\mathcal{Z}\{B^{k}\ e(k)\} = E(z)_{z=z/B}$$ **where B is any constant**

Example 1.6:
Given that the z-transform of e(k)=k is

$$E(z) = z/(z-1)^2,$$

then the z-transform of ke^{ak} is ;

$$E(z)|_{z \leftarrow z\epsilon^{-a}} = \left.\frac{z}{(z-1)^2}\right|_{z \leftarrow z\epsilon^{-a}}$$
$$= \frac{z\epsilon^{-a}}{(z\epsilon^{-a}-1)^2} = \frac{\epsilon^{a} z}{(z-\epsilon^{a})^2}$$

As a rule :

$$Z\{a^{k}\, f(k)\} = F(z)|_{z=\frac{z}{a}}$$

---End of Example-----------

5- Initial Value:

Property. Given that z-transform of e(k) is E(z), then;

$$e(0) = \lim_{z \to \infty} E(z)$$

Proof. Since;

$$E(z) = e(0) + e(1)z^{-1} + e(2)z^{-2} + \cdots$$

Then 2-18 is seen by inspection.

6- Final Value:

Property. Given that the z-transform of e(k) is E(z), then ;

$$\lim_{n \to \infty} e(n) = \lim_{z \to 1} (z - 1)E(z)$$

Provided that the left-side limit exist.

Proof. Consider the transform;

$$\begin{aligned} \mathcal{z}[e(k+1) - e(k)] &= \lim_{n \to \infty} \left[\sum_{k=0}^{n} e(k+1)z^{-k} - \sum_{k=0}^{n} e(k)z^{-k} \right] \\ &= \lim_{n \to \infty} [-e(0) + e(1)(1 - z^{-1}) + e(2)(z^{-1} - z^{-2}) + \cdots \\ &\quad + e(n)(z^{-n+1} - z^{-n}) + e(n+1)z^{-n}] \end{aligned}$$

Thus

$$\lim_{z \to 1} [\mathcal{z}[e(k+1) - e(k)]] = \lim_{n \to \infty} [e(n+1) - e(0)]$$

Also from the real translation property,

$$\begin{aligned} \mathcal{z}[e(k+1) - e(k)] &= z[E(z) - e(0)] - E(z) \\ &= (z - 1)E(z) - ze(0) \end{aligned}$$

Equating the two expressions above, we obtain

$$\lim_{n\to\infty} e(n) = \lim_{z\to 1}(z-1)E(z)$$

Provided that the left-side limit exists.

<u>Example 1.7:</u>

To illustrate the initial-value property and the final-value property, consider the z-transform of e(k)=1, whereas k=0,1,2, ... We have shown in example 2.2 that;

$$E(z) = \mathfrak{z}[1] = \frac{z}{z-1}$$

Applying initial-value property, we see that;

$$e(0) = \lim_{z\to\infty}\frac{z}{z-1} = \lim_{z\to\infty}\frac{1}{1-1/z} = 1$$

Since the final value of e(k) exist, we may apply the final-value property;

$$\lim_{k\to\infty} e(k) = \lim_{z\to 1}(z-1)E(z) = \lim_{z\to 1} z = 1$$

---End of Example-----------

The derived properties of the z-transform are listed in Table 1.2, which also includes additional properties. The notation e1(k) * e2(k) indicates convolution process (will be discussed later).

Table (1.2) PROPERTIES OF THE z-TRANSFORM

Sequence	Transform
$e(k)$	$E(z) = \sum_{k=0}^{\infty} e(k)z^{-k}$
$a_1 e_1(k) + a_2 e_2(k)$	$a_1 E_1(z) + a_2 E_2(z)$
$e(k-n)u(k-n); \quad n \geq 0$	$z^{-n}E(z)$
$e(k+n)u(k); \quad n \geq 1$	$z^n\left[E(z) - \sum_{k=0}^{n-1} e(k)z^{-k}\right]$
$\epsilon^{ak} e(k)$	$E(z\epsilon^{-a})$
$ke(k)$	$-z\frac{dE(z)}{dz}$
$e_1(k) * e_2(k)$	$E_1(z)E_2(z)$
$e_1(k) = \sum_{n=0}^{k} e(n)$	$E_1(z) = \frac{z}{z-1}E(z)$
Initial value: $e(0) = \lim_{z \to \infty} E(z)$	
Final value: $e(\infty) = \lim_{z \to 1} (z-1)E(z)$, if $e(\infty)$ exists	

1.5 Solution of Difference Equation by z-transform.

Consider the following nth-order difference equation, where it is assumed that {e(k)} is known.

$$m(k) + a_{n-1}m(k-1) + \cdots + a_0 m(k-n)$$
$$= b_n e(k) + b_{n-1}e(k-1) + \cdots + b_0 e(k-n)$$

The z-transform of equation above, which results from the use of the real translation property is;

$$M(z) + a_{n-1}z^{-1}M(z) + \cdots + a_0 z^{-n}M(z)$$
$$= b_n E(z) + b_{n-1}z^{-1}E(z) + \cdots + b_0 z^{-n}E(z)$$

Note that the z-transform has changed the difference equation to an algebraic equation in. Solving the expression above for M(z) ;

$$M(z) = \frac{b_n + b_{n-1}z^{-1} + \cdots + b_0 z^{-n}}{1 + a_{n-1}z^{-1} + \cdots + a_0 z^{-n}} E(z)$$

M(k) can be found by taking the inverse z-transform (will be discussed later).

<u>Example 1.8:</u>

Consider the difference equation ;

$$m(k) = e(k) - e(k-1) - m(k-1)$$

The z-transform of this equation, obtained via the real translation property, is;

$$M(z) = E(z) - z^{-1}E(z) - z^{-1}M(z)$$

Or

$$M(z) = \frac{z-1}{z+1}E(z)$$

Since

$$e(k) = \begin{cases} 1, & k \text{ even} \\ 0, & k \text{ odd} \end{cases}$$

We see that

$$E(z) = 1 + z^{-2} + z^{-4} + \cdots = \frac{1}{1-z^{-2}} = \frac{z^2}{z^2-1} = \frac{z^2}{(z-1)(z+1)}$$

Thus

$$M(z) = \frac{z-1}{z+1}\,\frac{z^2}{(z-1)(z+1)} = \frac{z^2}{z^2+2z+1}$$

We can expand M(z) into a power series by dividing the numerator of M(z) by its denominator so as to obtain;

$$\begin{array}{r l}
 & 1 - 2z^{-1} + 3z^{-2} - 4z^{-3} + \cdots \\
z^2 + 2z + 1 \,\big)\!\!\! & \overline{z^2} \\
 & \underline{z^2 + 2z + 1} \\
 & \quad -2z - 1 \\
 & \quad \underline{-2z - 4 - 2z^{-1}} \\
 & \qquad\qquad 3 + 2z^{-1} \\
 & \qquad\qquad \underline{3 + 6z^{-1} + 3z^{-2}} \\
 & \qquad\qquad\quad -4z^{-1} - 3z^{-2} \\
 & \qquad\qquad\qquad \cdots
\end{array}$$

Therefore;

$$M(z) = 1 - 2z^{-1} + 3z^{-2} - 4z^{-3} + \cdots$$

And values of m(k) are to be obtained using the inverse z-transform (which will be discussed in the coming sections).

---End of Example-----------

1.6 The Inverse z-transform:

In order for the z-transform technique to be a feasible approach in the solution of difference equation, methods for determining the inverse z-transform are required.

1- power Series Method:

The power series method for finding the inverse z-transform of a function E(z) which is expressed as the ratio of two polynomials in z involves dividing the denominator of E(z) into the numerator such that a power series of the form

$$E(z) = e_0 + e_1 z^{-1} + e_2 z^{-2} + \cdots$$

Is obtained. From the definition of z-transform, it can be seen that the values of e(k) are simply the coefficients in the power series.

<u>Example 1.9:</u>
It is desired to find the values of e(k) for E(z) given by the expression;

$$E(z) = \frac{z}{z^2 - 3z + 2}$$

Using long division, we obtain;

$$\begin{array}{r l}
 & z^{-1} + 3z^{-2} + 7z^{-3} + 15z^{-4} + \cdots \\
z^2 - 3z + 2 \,\big) & z \\
 & \underline{z - 3 + 2z^{-1}} \\
 & \quad 3 - 2z^{-1} \\
 & \quad \underline{3 - 9z^{-1} + 6z^{-2}} \\
 & \qquad 7z^{-1} - 6z^{-2} \\
 & \qquad \underline{7z^{-1} - 21z^{-2} + 14z^{-3}} \\
 & \qquad\qquad 15z^{-2} - 14z^{-3} + \cdots \\
 & \qquad\qquad \cdots\cdots\cdots
\end{array}$$

And therefore;

$$
\begin{aligned}
e(0) &= 0 \qquad & e(4) &= 15 \\
e(1) &= 1 & &\cdots \\
e(2) &= 3 & e(k) &= 2^k - 1 \\
e(3) &= 7 & &\cdots
\end{aligned}
$$

In this particular case, the general expression for e(k) as a function of k (i.e., $e(k) = 2^k - 1$) can be recognized. In general this cannot be done using the power series.

---End of Example-----------

2- Partial-Fraction Expansion Method:

A function E(z) can be expanded in patial fractions and then tables of known z-transform is given in Table 1-3, and a table of z-transform based on sampled time function is given in "z-transform Tables".

Before proceeding with an example of the partial-fraction expansion method, consider the function;

$$E(z) = \frac{z}{z-a} = 1 + az^{-1} + a^2 z^{-2} + a^3 z^{-3} + \cdots$$

Examining of the power series inicates that;

$$\mathfrak{z}^{-1}\left[\frac{z}{z-a}\right] = a^k$$

Where $\mathfrak{z}^{-1}[\cdot]$ indicates the inverse z-transfor. This particular function is perhaps the most commen z-transform, since the sequence $\{a^k\}$ is exponential in nature.

Table (1.3) z-TRANSFORMS

Sequence	z-Transform
$\delta(k-n)$	z^{-n}
1	$\dfrac{z}{z-1}$
k	$\dfrac{z}{(z-1)^2}$
k^2	$\dfrac{z(z+1)}{(z-1)^3}$
a^k	$\dfrac{z}{z-a}$
ka^k	$\dfrac{az}{(z-a)^2}$
$\sin ak$	$\dfrac{z \sin a}{z^2 - 2z \cos a + 1}$
$\cos ak$	$\dfrac{z(z - \cos a)}{z^2 - 2z \cos a + 1}$
$a^k \sin bk$	$\dfrac{az \sin b}{z^2 - 2az \cos b + a^2}$
$a^k \cos bk$	$\dfrac{z^2 - az \cos b}{z^2 - 2az \cos b + a^2}$

Now, if the factor z appears at the numerator, the patial fraction should be performed on E(z)/z , which will result in the terms of E(z) as in the table of “z-transform tables”.

Example 1.10:

Consider the function E(z) given below;

$$E(z) = \frac{z}{(z-1)(z-2)}$$

Hence

$$\frac{E(z)}{z} = \frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$$

Then

$$\mathcal{Z}^{-1}[E(z)] = \mathcal{Z}^{-1}\left[\frac{-z}{z-1}\right] + \mathcal{Z}^{-1}\left[\frac{z}{z-2}\right]$$

From equation (2-26) or Table 2-3, the value of e(k) is;

$$e(k) = -1 + 2^k$$

Which is the same value as that in example 2.10.
Consider next the function

$$E_1(z) = z^{-1}E(z) = \frac{1}{(z-1)(z-2)}$$

From the real translation property (2-14), $e_1(k)$ is given by;

$$\begin{aligned} e_1(k) &= \mathcal{Z}^{-1}[z^{-1}E(z)] = e(k-1)u(k-1) \\ &= [-1 + 2^{(k-1)}]u(k-1) \\ &= \begin{cases} 0, & k = 0 \\ -1 + 2^{(k-1)}, & k \geq 1 \end{cases} \end{aligned}$$

The inverse can also be found by partial-fraction expansion.

$$\frac{E_1(z)}{z} = \frac{1}{z(z-1)(z-2)} = \frac{\frac{1}{2}}{z} + \frac{-1}{z-1} + \frac{\frac{1}{2}}{z-2}$$

Or,

$$E_1(z) = \frac{1}{2} + \frac{-z}{z-1} + \frac{(\frac{1}{2})z}{z-2}$$

Thus

$$e_1(k) = a - 1 + (\tfrac{1}{2})(2)^k = a - 1 + (2)^{k-1}$$

Where

$$a = \begin{cases} \frac{1}{2}, & k = 0 \\ 0, & k \geq 1 \end{cases}$$

Since from Table 2-3;

$$\mathfrak{z}^{-1}\left[\frac{1}{2}\right] = \frac{1}{2}\delta(k) = \begin{cases} \frac{1}{2}, & k = 0 \\ 0, & k \geq 1 \end{cases}$$

Hence the two procedures yield the same result for $e_1(k)$.

---End of Example-----------

Thus far we have considerd the inverse transformation by partial fractions only for functions that have real poles. The same partial-fraction procedure applies for complex poles;however, the resulting inverse transform contains complex functions. Of course, the sum of these functions is real.

First, consider the real function;

$$y(k) = A\epsilon^{akT}\cos(bkT + \theta) = \frac{A\epsilon^{akT}}{2}[\epsilon^{jbkT}\epsilon^{j\theta} + \epsilon^{-jbkT}\epsilon^{-j\theta}]$$

$$= \frac{A}{2}[\epsilon^{(aT + jbT)k}\epsilon^{j\theta} + \epsilon^{(aT - jbT)k}\epsilon^{-j\theta}]$$

Where a and b are real. Euler's relation, given by;

$$\cos x = \frac{\epsilon^{jx} + \epsilon^{-jx}}{2}$$

The z-transform of this function is given by (from "z-transform tables" appendix) ;

$$Y(z) = \frac{A}{2}\left[\frac{\epsilon^{j\theta} z}{z - \epsilon^{aT + jbT}} + \frac{\epsilon^{-j\theta} z}{z - \epsilon^{aT - jbT}}\right]$$

$$= \frac{(A\epsilon^{j\theta}/2)z}{z - \epsilon^{aT + jbT}} + \frac{(A\epsilon^{-j\theta}/2)z}{z - \epsilon^{aT - jbT}} = \frac{k_1 z}{z - p_1} + \frac{k_1^* z}{z - p_1^*}$$

Where the asterisk indicates the "complex conjugate".
Hence, given the partial-fraction coefficient k1and p1 in equation above, we can solve for the discrete-time function using the following relationship;

$$p_1 = \epsilon^{aT}\epsilon^{jbT} = \epsilon^{aT}\angle bT \Rightarrow aT = \ln|p_1|; \qquad bT = \arg p_1$$

And

$$k_1 = \frac{A\epsilon^{j\theta}}{2} = \frac{A}{2}\angle\theta \Rightarrow A = 2|k_1|; \qquad \theta = \arg k_1$$

Hence we calculate aT and bT from the poles, and A and θ from partial-fraction expansion. We can express the inverse transform as the sinusoid of eq (2-27). An illustrative example is given below;

Example 1.11:
We find the inverse z-transform of the function;

$$Y(z) = \frac{-3.894z}{z^2 + 0.6065} = \frac{-3.894z}{(z - j0.7788)(z + j0.7788)}$$

$$= \frac{k_1 z}{z - j0.7788} + \frac{k_1^* z}{z + j0.7788}$$

Dividing both sides by z, we calculate k1 :

$$k_1 = (z - j0.7788)\left[\frac{-3.894}{(z - j0.7788)(z + j0.7788)}\right]_{z = j0.7788}$$

$$= \left.\frac{-3.894}{z + j0.7788}\right|_{z = j0.7788} = \frac{-3.894}{2(j0.7788)} = 2.5\angle 90^\circ$$

From eq (2-29) and (2-30), with p1=j 0.7788,

$$aT = \ln|p_1| = \ln(0.7788) = -0.250; \qquad bT = \arg p_1 = \pi/2$$
$$A = 2|k_1| = 2(2.5) = 5; \qquad \theta = \arg k_1 = \pi/2$$

Hence, from eq (2-27),

$$y(k) = A\epsilon^{akT}\cos(bkT + \theta)$$
$$= 5\epsilon^{-0.25k}\cos\left(\frac{\pi}{2}k + \frac{\pi}{2}\right) = -5\epsilon^{-0.25k}\sin\frac{\pi}{2}k$$

---End of Example-----------

3- Inversion-Formula Method:

Perhaps the most general technique for obtaining the inverse of a z-transform is the inversion integral. This integral, derived via complex variable theory, is;

$$e(k) = \frac{1}{2\pi j}\oint_{\Gamma} E(z)z^{k-1}\,dz, \qquad j = \sqrt{-1}$$

This expression is the line integral in the z-plane along the closed path Γ, where Γ is any path that encloses all the finite poles of $E(z)z^{k-1}$.

Using the theorem of residues, we can evaluate the integral in equation above via the expression;

$$e(k) = \sum_{\substack{\text{at poles} \\ \text{of } [E(z)z^{k-1}]}} [\text{residues of } E(z)z^{k-1}]$$

If the function $E(z)z^{k-1}$ has a simple pole at z=a, the residue is evaluated as ;

$$(\text{residue})_{z=a} = (z-a)E(z)z^{k-1}\Big|_{z=a}$$

For a pole of order m at z=a, the residue is calculated using the expression ;

$$(\text{residue})_{z=a} = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}[(z-a)^m E(z)z^{k-1}]\Big|_{z=a}$$

Example 1.12:

Consider the function E(z) ;

$$E(z) = \frac{z}{(z-1)(z-2)}$$

Substituting this expression into eq (2-32) and (2-33) yields;

$$e(k) = \frac{z^k}{z-2}\Big|_{z=1} + \frac{z^k}{z-1}\Big|_{z=2} = -1 + 2^k$$

And the result is seen to be the same as that obtained in the previous examples. As in previous examples, let ;

$$E_1(z) = z^{-1}E(z) = \frac{1}{(z-1)(z-2)}$$

Then, from the real translation property,

$$e_1(k) = e(k-1)u(k-1) = [-1 + 2^{(k-1)}]u(k-1)$$

We can also find $e_1(k)$ by the inversion formula. In eq (2-32),

$$E_1(z)z^{k-1} = \frac{z^{k-1}}{(z-1)(z-2)}$$

This function has a pole at z=0 only for k=0, and thus;

$$e_1(0) = \sum_{\text{at } z=0,1,2}\left[\text{residues of } \frac{1}{z(z-1)(z-2)}\right]$$
$$= \tfrac{1}{2} - 1 + \tfrac{1}{2} = 0$$

The values of $e_1(k)$ for $k \geq 1$ is obtained directly, and is left as an exercise !!!! (Home work)
--End of Example-----------

Example 1.13:

The function E(z) below has a single pole of order **2** at z=1.

$$E(z) = \frac{z}{(z-1)^2}$$

The inverse transform obtained using eq (2-34) is

$$e(k) = \frac{1}{(2-1)!}\frac{d^{2-1}}{dz^{2-1}}\left[(z-1)^2\left(\frac{z}{(z-1)^2}\right)z^{k-1}\right]\Bigg|_{z=1}$$

$$= \frac{d}{dz}(z^k)\Big|_{z=1}$$

$$= kz^{k-1}\Big|_{z=1}$$

$$= k$$

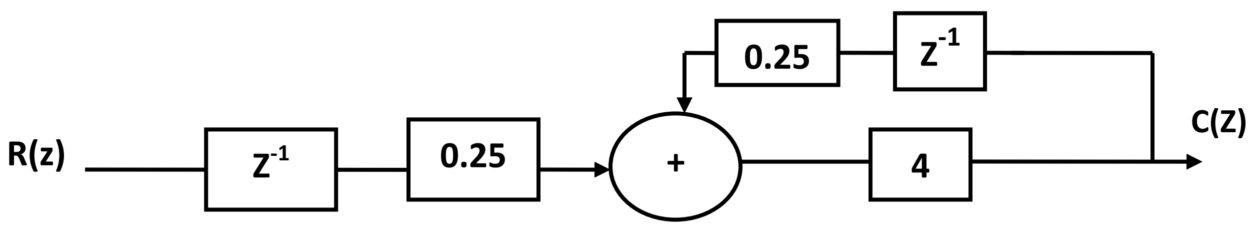
---End of Example-----------

1.7 Application of difference equation and their solution

The use of z-transform together with the difference equation representation of a discrete system may be shown by the following examles;

Example 1.14:

For the discrete system shown below, find c(k) with r(k)=u(k-2)+u(k)



First of all, we should obtain R(Z) by taking the Z-T of r(k),

$$R(Z) = Z^{-2}\frac{Z}{Z-1} + \frac{Z}{Z-1} = \frac{Z^{-1}}{Z-1} + \frac{Z}{Z-1}$$

then;

$$R(Z) = \frac{Z^{-1}+Z}{Z-1} \qquad \text{--------(1)}$$

Secondly; relation of C(Z) and R(Z) must be found from the system diagram;

$C(Z) = 4\,[0.25\,Z^{-1}\,R(Z) + 0.25\,Z^{-1}\,C(Z)\,] \; = Z^{-1}\,R(Z)\; + Z^{-1}\,C(Z)$

$C(Z)\,[1 - Z^{-1}] \; = \; Z^{-1}\,R(Z)$
Then;

$$C(Z) = \frac{Z^{-1}\,R(Z)}{1-Z^{-1}} \quad \text{and finaly} \quad C(Z) = \frac{R(Z)}{Z-1} \qquad \text{------(2)}$$

Thirdly; substituting eq.1 into eq.2;

$$C(Z) = \frac{1}{Z-1} \quad \frac{Z^{-1}+Z}{Z-1} \; = \; \frac{Z^{-1}+Z}{(Z-1)^2}$$

then;

$$C(Z) = \frac{Z^{-1}}{(Z-1)^2} + \frac{Z}{(Z-1)^2} \qquad \text{---------(3)}$$

Finally, by taking the inverse Z-T of C(Z) we obtain c(k);

$$c(k) = (\,k-2)\,U(k-2) + \; k\,U(k)$$

end of Example

Excersices:

1- Without using long-division, partial fraction, and inversion formula, find f(k) for;

$$\boldsymbol{F(z)=\frac{(z-4)\quad(z^2-8z+16)+4\,z^4+16z^3+z^3(z^2-8z+16)}{z^2(z-4)\quad(z^2-8z+16)}}$$

2- Find Z-transform of δ(k) , δ(k-4), and δ(k+2).

3- using long division, find f(k) for

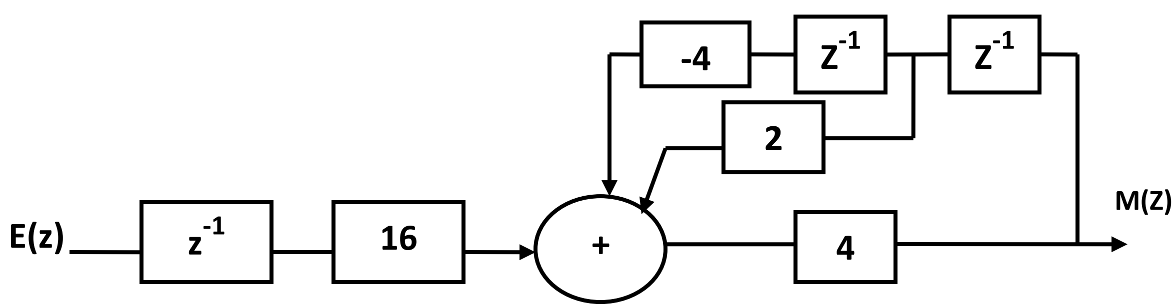
$$\mathbf{F(z)=\frac{z^2}{z^2+1.2\,z+0.2}}$$

4- **Without** using partial fraction, long division, and residue methods find m(k) for:

$$M(z)=\frac{16+2z^2}{z^2-4z+4}$$

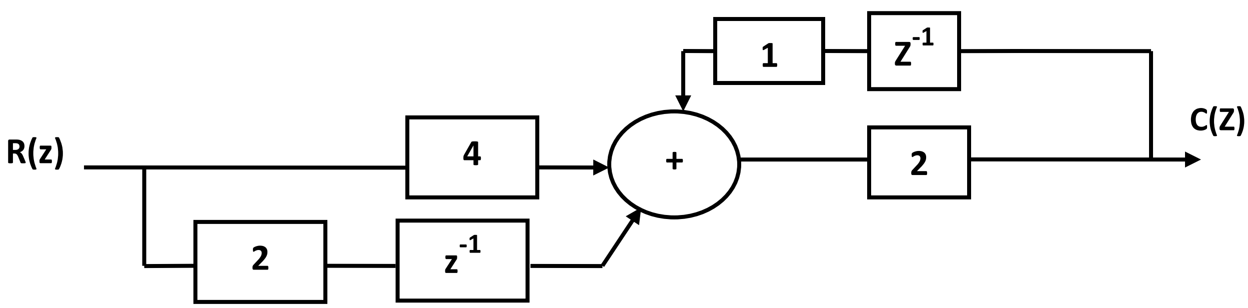
5 - Find m(k) for the below discrete system, knowing that

e(k) =δ(k-2)+ 4^k u(k)+4 x $4^{(k-1)}$ u(k-1)



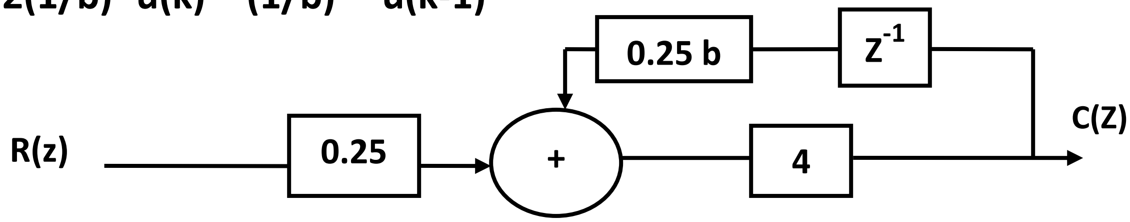
6- Find **c(k)** for the below discrete system, knowing that ;

R(k) = 2^k u(k) + $2^{(k-2)}$ u(k-2)



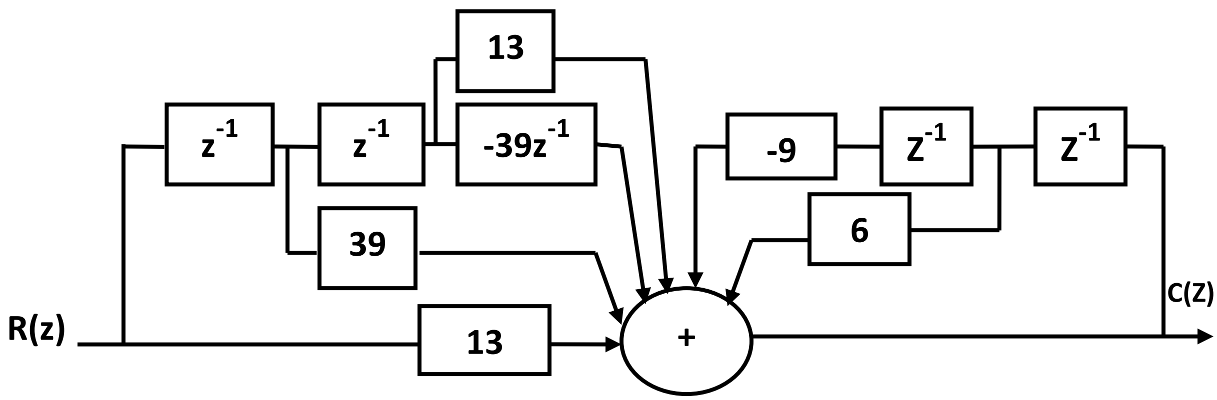
7- Find c(k) for the below discrete system, knowing that ;

$\mathbf{r(k) = 2(1/b)^{k}\, u(k) - (1/b)^{k-1}\, u(k-1)}$



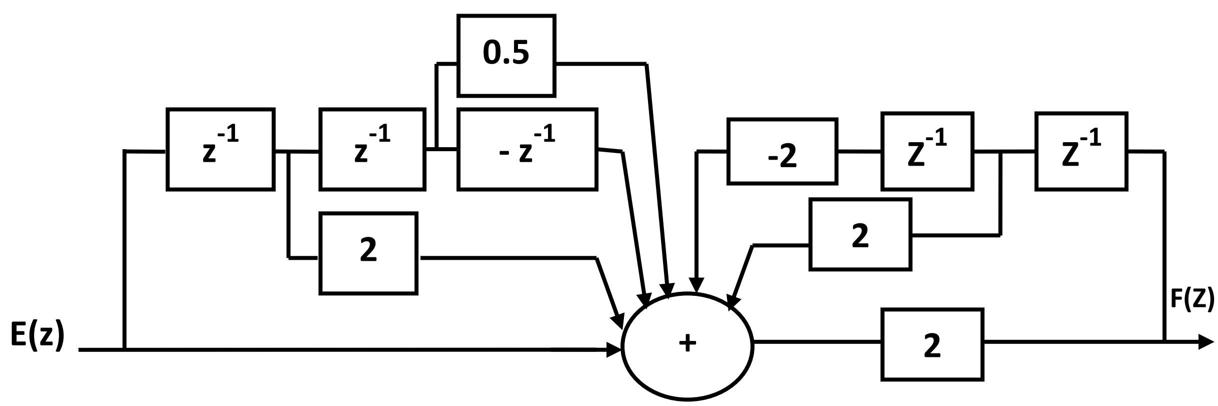
8- Find c(k) for the below discrete system, knowing that

$\mathbf{r(k) = \delta(k+2) + 3^{k}\, u(k) + 4 \times 4^{(k+1)}\, \delta\,(k+1)}$



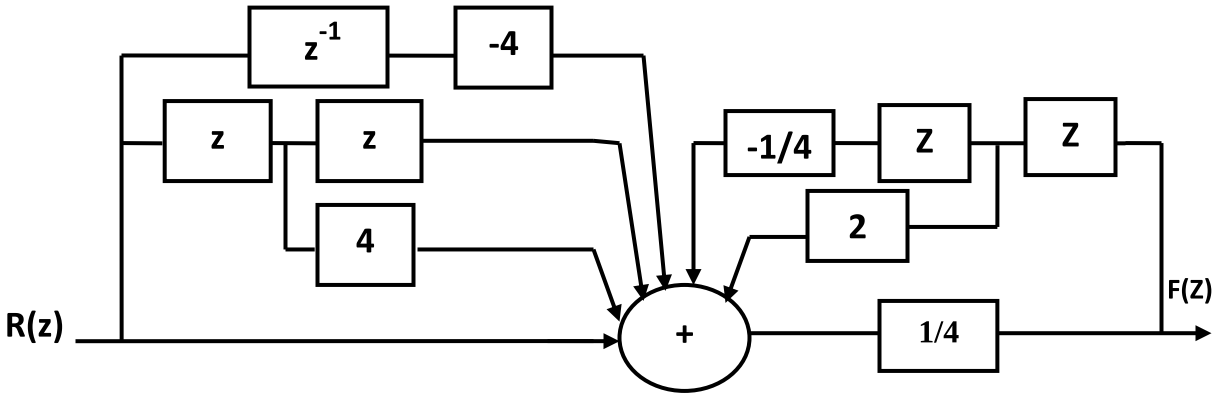
9- Find f(k) for the below discrete system, knowing that

$\mathbf{e(k) = 2 \times 2^{k-1}\;\; u(k) + 4 \times 4^{(k+1)}\, \delta\,(k+2)}$



10- Find f(k) for the below discrete system, knowing that

$r(k) = 2 \times 2^{k-1}\ u(k) + 4 \times 4^{(k+1)}\ \delta\ (k+2)$



11- Find f(k) for the below discrete system, knowing that

$r(k) = 3 \times 0.5^{k-1}\ u(k-1) + 4 \times 4^{(k+1)}\ \delta\ (k+2)$