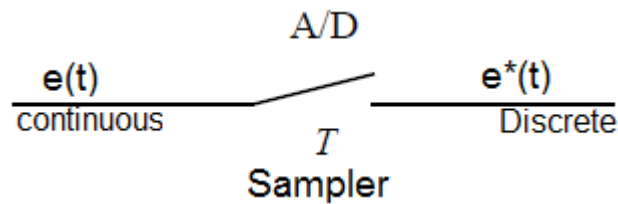




## Sampling and Re-Construction

### 2.1 Introduction

Consider a continuous signal which is sampled, as shown below, in which continuous signal  $e(t)$  is sampled to produce a discrete signal which we will designate  $e^*(t)$ .



### 2.2 Sampled Data Control Systems

In general, it is undesirable to apply a signal in sampled form, such as a train of narrow rectangular pulses, as shown in Figure (2.1) to a plant because of the high frequency components generated. Therefore, a data reconstruction device (DATA HOLD) is inserted into the system directly following the sampler. The purpose of the data hold is to reconstruct the sampled signal into a form that closely resembles the signal before sampling. The simplest data reconstruction device is the ZERO-order hold.

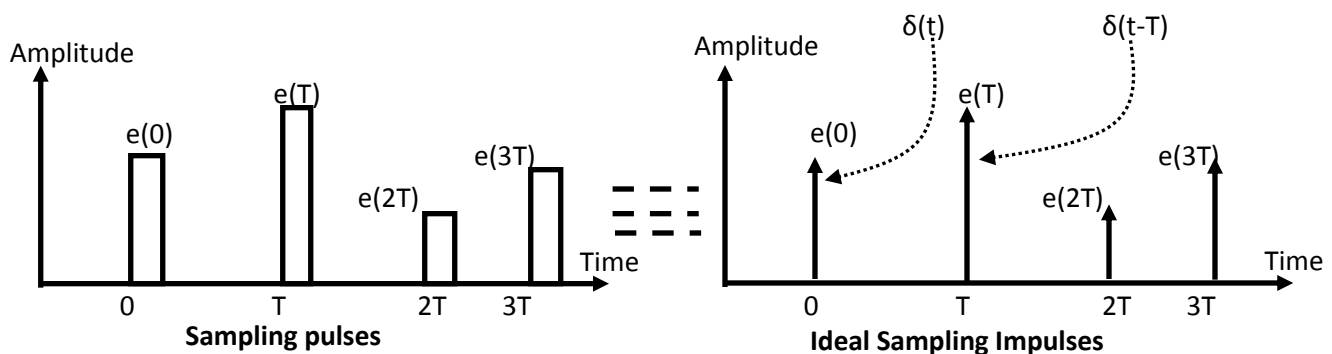


Figure 2.1 Sampled signal in pulse and impulse forms



The operation of the sampler and Zero Order Hold ZOH combination is described by the signals shown in Figure 2.2. the zero order hold clamps the output signal to a value equal to that of the input signal at the sampling instant.

The sampler and zero order hold can be shown as blocks as in Figure 2.3. the signal  $\bar{e}(t)$  can be expressed as;

$$\bar{e}(t) = e(0)[u(t) - u(t - T)] + e(T)[u(t - T) - u(t - 2T)] + e(2T) - u(t - 3T) + \dots$$

where  $u(t)$  is the unit step function. The Laplace transform of  $\bar{e}(t)$  is  $\bar{E}(s)$  and given by;

$$\begin{aligned} \bar{E}(s) &= e(0) \left[ \frac{1}{s} - \frac{e^{-Ts}}{s} \right] + e(T) \left[ \frac{e^{-Ts}}{s} - \frac{e^{-2Ts}}{s} \right] + e(2T) \left[ \frac{e^{-2Ts}}{s} - \frac{e^{-3Ts}}{s} \right] + \dots \\ &= \left[ \frac{1 - e^{-Ts}}{s} \right] [e(0) + e(T)e^{-Ts} + e(2T)e^{-2Ts} + e(3T)e^{-3Ts} + \dots] \\ &= \left[ \sum_{n=0}^{\infty} e(nT) e^{-nTs} \right] \left[ \frac{1 - e^{-Ts}}{s} \right] \end{aligned}$$

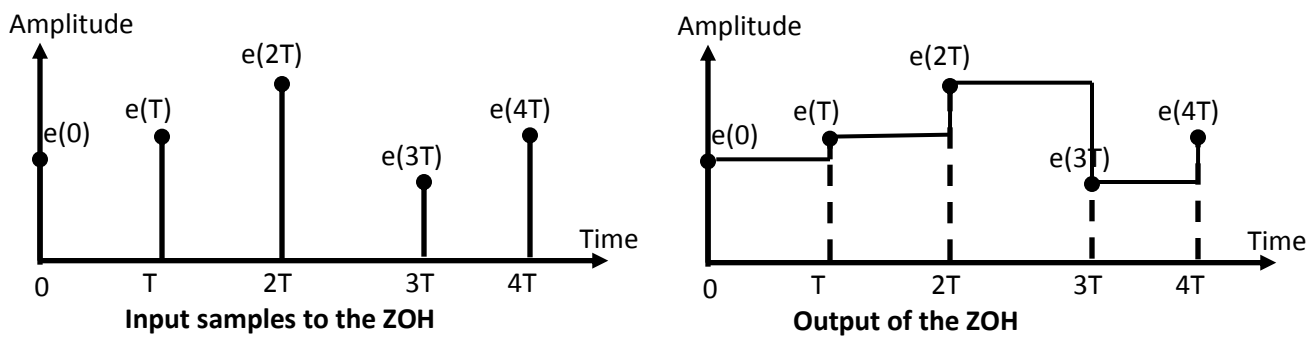


Figure 2.2 ZOH input and output signals

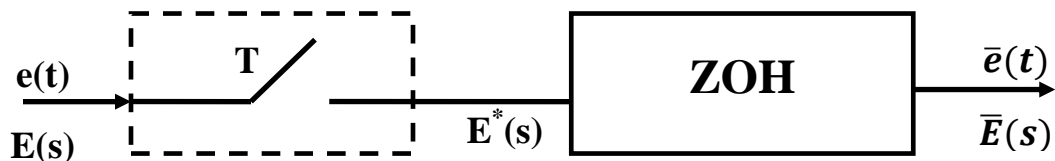


Figure 2.3 Sampler and ZOH



Now since  $E^*(s)$  represent the starred transform of the sampled signal (Laplace transform of the sampled signal) and is written as;

$$E^*(s) = \sum_{n=0}^{\infty} e(nT) e^{-nTs}$$

Also  $G_{h0}(s)$  is the ZOH s- domain equation and is written as;

$$G_{h0}(s) = \left[ \frac{1 - e^{-Ts}}{s} \right]$$

Therefore,  $\bar{E}(s)$  may be written as;

$$\bar{E}(s) = E^*(s) G_{h0}(s)$$

Using this equation the sampling and reconstruction process may be represented as shown in Figure 2.4. It is to be emphasized that  $E^*(s)$  does not appear in the physical system but appears as a result of factoring.

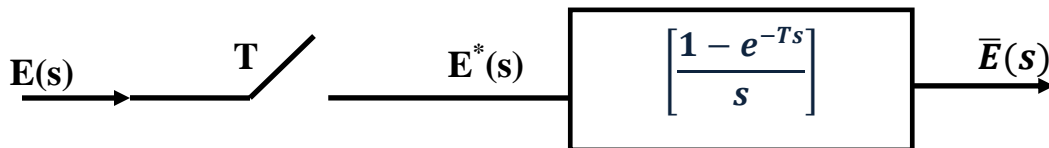


Figure 2.4 Representation of sampler and data hold

## 2.3 The Ideal Sampler

Using Figure 2.1 previously shown, the sampled time representation of the ideal sampler output is;

$$e^*(t) = e(0)\delta(t) + e(T)\delta(t - T) + e(2T)\delta(t - 2T) + \dots$$

where  $\delta(t)$  is the unit impulse function occurring at  $t=0$ . Then  $e^*(t)$  is a train of impulse functions whose weights are equal to the values of signal at the instants of sampling.



Thus,  $e^*(t)$  can be represented as shown in Figure 2.5, since the impulse function has infinite amplitude at the instant it occurs. Note again that  $e^*(t)$  is not a physical signal.

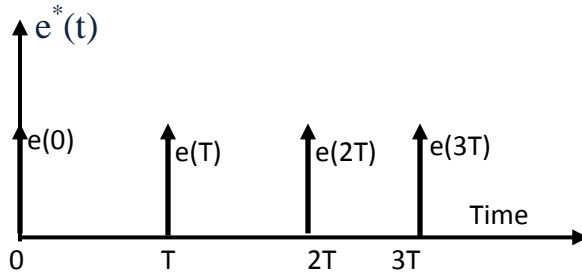


Figure 2.5 Representation of  $e^*(t)$  using impulse functions

## 2.4 Frequency Spectrum of the Sampled Signal

We have a method of mathematically representing continuous functions  $e(t)$  which have been sampled. The question posed in this section is, what is the frequency spectrum of sampled signal  $e^*(t)$ ? We will first find a Fourier series representation of the impulse train, use this in the expression for  $e^*(t)$ , take the Laplace transform of  $e^*(t)$ , which is  $R^*(s)$ , then use the fact that  $R^*(j\omega) = R^*(s)_{\text{at } s=j\omega}$ .

Now, since the complete impulse train

$$\sum_{k=-\infty}^{k=\infty} \delta(t - kT)$$

is periodic with period  $T$ , it can be represented by a Fourier Series;

$$\sum_{k=-\infty}^{k=\infty} \delta(t - kT) = \sum_{n=-\infty}^{n=\infty} C_n e^{j\left(\frac{2\pi n}{T}\right)t}$$

where the Fourier coefficients are given by;

$$C_n = \frac{1}{T} \int_{-T/2}^{T/2} \sum_{k=-\infty}^{k=\infty} \delta(t - kT) e^{-j\left(\frac{2\pi n}{T}\right)t}$$



According to the sieving property of the impulse function (and the only impulse within the integration limits is at  $t = 0$ ) yields a very simple result,

$$C_n = \frac{1}{T} \left[ e^{-j\left(\frac{2\pi n}{T}\right) \cdot 0} \right] = \frac{1}{T}$$

Substituting the Fourier coefficients  $C_n$  into the Fourier series, so

$$\sum_{k=-\infty}^{k=\infty} \delta(t - kT) = \sum_{n=-\infty}^{n=\infty} \frac{1}{T} e^{j\left(\frac{2\pi n}{T}\right)t} = \frac{1}{T} \sum_{n=-\infty}^{n=\infty} e^{j(n\omega_s)t}$$

where we have used the relation that sampling frequency  $\omega_s = 2\pi/T$ . Using the Fourier series expression for the impulse train, we can express sampled signal  $E^*(t)$  as;

$$e^*(t) = \frac{1}{T} e(t) \sum_{n=-\infty}^{\infty} e^{j(n\omega_s)t}$$

The Laplace transform  $E^*(s)$  of sampled signal  $e^*(t)$  is;

$$\begin{aligned} E^*(s) &= \int_{-\infty}^{\infty} \frac{1}{T} e(t) \sum_{n=-\infty}^{\infty} e^{j(n\omega_s)t} e^{-st} dt \\ &= \frac{1}{T} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e(t) e^{-(s-jn\omega_s)t} dt \end{aligned}$$

And applying exponential property;

$$\frac{1}{T} \sum_{n=-\infty}^{\infty} E(s - jn\omega_s)$$

Substituting  $s=j\omega$ ,

$$E^*(j\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} E(j\omega - jn\omega_s)$$



This equation will reflect that there are an infinite number of repeated spectra in the sampled signal, and they are repeated every  $\omega_s = 2\pi/T$  along the frequency axis. Furthermore, there are two case resulting from this sampling as shown in Figure 2.6.

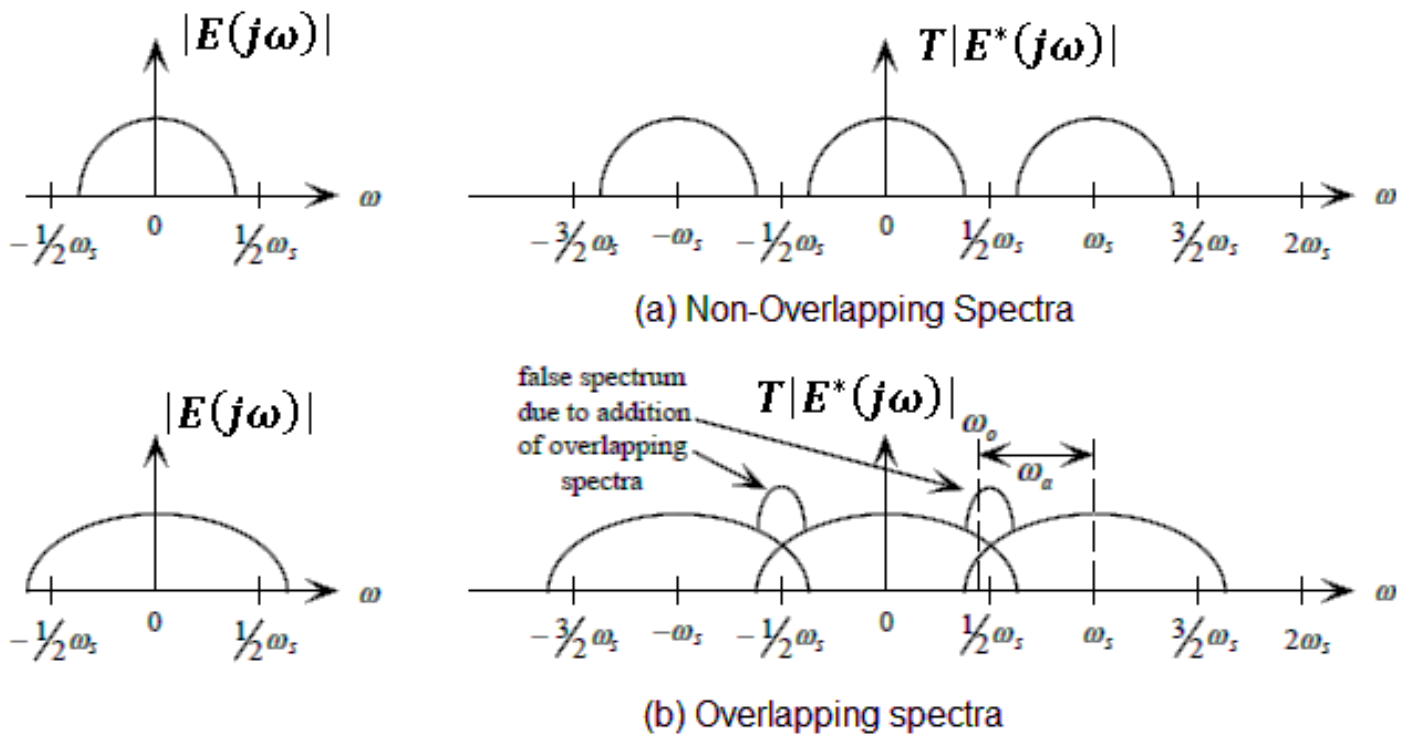


Figure 2.6 Frequency spectra of two signals

Aliasing means that two different sinusoids have identical samples, and we cannot distinguish between them from their samples. When experimental data are to be sampled, it is essential that an "antialiasing" analog filter be used before sampling to filter out frequencies above one-half the sampling frequency (called the Nyquist frequency). Otherwise, frequencies above the Nyquist frequency in the continuous signal will erroneously appear as lower frequencies after sampling.



## 2.5 Obtaining Z-transform from Laplace form using Residue Method:

Let  $F(s)$  is the Laplace transform of a certain signal, then the Z-transform of this signal is;

$$F(Z) = \sum \text{Residues of } F(S) \frac{Z}{Z - e^{ST}} \text{ at poles of } F(S)$$

$$\text{Res.}_{s=a} = \frac{1}{(m-1)!} \lim_{S \rightarrow a} \frac{d^{m-1}}{ds^{m-1}} \left[ (S-a)^m F(S) \frac{Z}{Z - e^{ST}} \right]$$

### Example 2.1:

If  $F(s)=1/s$  and  $T=1$  sec, find  $F(z)$ .

There is one pole at  $s=0$ . One residue is;

$$\text{Res.}_{s=0} = \lim_{s \rightarrow 0} s \left( \frac{1}{s} \cdot \frac{Z}{Z - e^{sT}} \right) = \frac{Z}{Z-1}$$

### Example 2.2:

If  $F(s) = \frac{e^{-2s}}{s(s+2)}$  and  $T=1$ , find  $F(z)$ .

Remembering that  $z=e^s$ , therefore;

$$F(z) = z^{-2} Z \left\{ \frac{1}{s(s+2)} \right\} = z^{-2} \left[ \frac{1}{(s+2)} \frac{z}{z - e^s} \right]_{s=0} + z^{-2} \left[ \frac{1}{s} \frac{z}{z - e^s} \right]_{s=-2}$$

$$F(z) = \frac{1}{2} \frac{z^{-1}}{z-1} - \frac{1}{2} \frac{z^{-1}}{z - e^{-2}}$$

### Example 2.3:

If  $F(s) = \frac{1}{s(s+1)^2}$  and  $T=0.1$  sec, find  $F(z)$ .

$$F(z) = \left[ \frac{1}{(S+1)^2} \frac{z}{z - e^{0.1s}} \right]_{s=0} + \frac{d}{dz} \left[ \frac{1}{s} \frac{z}{z - e^{0.1s}} \right]_{s=-1}$$



$$F(z) = \left[ \frac{z}{z-1} \right] + z \frac{d}{dz} \left[ s^{-1} (z - e^{0.1s})^{-1} \right]_{s=-1}$$

$$F(z) = \left[ \frac{z}{z-1} \right] + z \left[ 0.1 s^{-1} e^{0.1s} (z - e^{0.1s})^{-2} - s^{-2} (z - e^{0.1s})^{-1} \right]_{s=-1}$$

$$F(z) = \left[ \frac{z}{z-1} \right] + z \left[ -0.1 e^{-0.1} (z - e^{-0.1})^{-2} - (z - e^{-0.1})^{-1} \right]$$

$$F(z) = \frac{z}{z-1} - \frac{0.09z}{(z-0.9)^2} - \frac{z}{(z-0.9)}$$

### Class Work:

Find Z-transform for the below S-domain function:

$$F(s) = \frac{1}{s^2(s+0.5)} \quad \text{Answer is } F(z) = \frac{z}{z-1} - \frac{z}{z-e^{-T}}$$

### Home Work:

Find the Z-transform of

$$1- F(s) = \frac{1}{s^2} \quad \text{Answer is } F(z) = \frac{zT}{(z-1)^2}$$

$$2- F(s) = \frac{1}{(s+L)^2} \quad \text{Answer is } F(z) = \frac{zTe^{-LT}}{(z-e^{-LT})^2}$$

## 2.6 Zero-Order Hold (ZOH):

Consider the figure below. Here we assume is approximately constant within the sampling interval at a value equal to that of the function preceding sampling instant. Therefore the Zero Order hold ;



$$e_n(t) = e(nT), \quad \text{for } nT \leq t < (n+1)T$$





For unit impulse function  $\delta(t)$  as the input to the ZOH;

$$e_o(t) = u(t) - u(t - T)$$

as shown in Figure 2.7

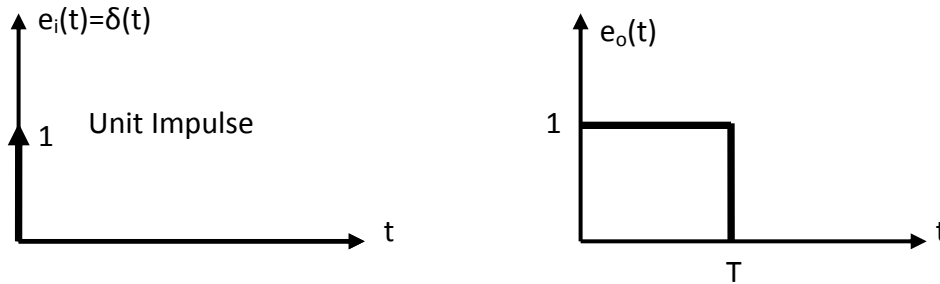


Figure 2.7 Input and output of signals of the ZOH

Taking Laplace transform;

$$E_o(s) = \frac{1}{s} - \frac{e^{-Ts}}{s}$$

Now, since  $E_i(s)=1$  (Laplace transform of  $\delta(t)$ ), then the transfer function of the ZOH may be written as;

$$G_{ho}(s) = \frac{E_o(s)}{E_i(s)} = \frac{1 - e^{-sT}}{s}$$

To derive and plot the frequency response of the ZOH, consider the equation developments below;

$$\begin{aligned} G_{ho}(j\omega) &= \frac{1 - e^{-j\omega T}}{j\omega} e^{j\omega T/2} e^{-j\omega T/2} = \frac{2e^{-j\omega T/2}}{\omega} \left[ \frac{e^{j\omega T/2} - e^{-j\omega T/2}}{2j} \right] \\ &= T \frac{\sin(\omega T/2)}{\omega T/2} e^{-j\omega T/2} \end{aligned}$$



Now since;  $\frac{\omega T}{2} = \frac{\omega}{2} \left( \frac{2\pi}{\omega_s} \right) = \frac{\pi\omega}{\omega_s}$  then ;

$$G_{h0}(j\omega) = T \frac{\sin(\pi\omega/\omega_s)}{\pi\omega/\omega_s} e^{-j\pi\omega/\omega_s}$$

Extracting the magnitude and phase of the transfer function ;

$$|G_{h0}(j\omega)| = T \left| \frac{\sin(\pi\omega/\omega_s)}{\pi\omega/\omega_s} \right|$$

and ;

$$\text{phase of } G_{h0}(j\omega) = -\frac{\pi\omega}{\omega_s} + \theta$$

$$\text{where } \theta = \begin{cases} 0 & \text{if } \sin\left(\frac{\pi\omega}{\omega_s}\right) > 0 \\ \pi & \text{if } \sin\left(\frac{\pi\omega}{\omega_s}\right) < 0 \end{cases}$$

The amplitude and phase plots for the ZOH transfer function is shown in Figure 2.8.

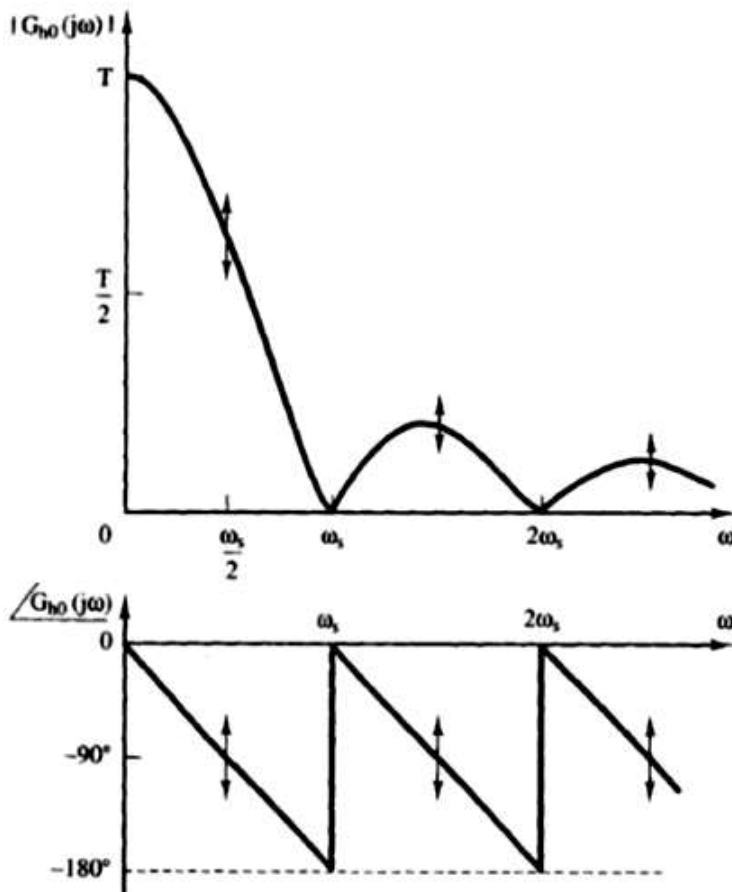


Figure 2.8 Frequency response of the ZOH



As an example assuming that the input to the sampler is sinusoid of frequency  $\omega_1$  which is less than  $\omega_s/2$ . Recall that the Fourier transform of  $e(t)=2\cos\omega_1t$  is given by;

$$E(j\omega) = \mathcal{F}\{2\cos(\omega_1 t)\} = \delta(\omega - \omega_1) + \delta(\omega + \omega_1)$$

Therefore, the spectrum of this signal is then two unit impulse functions as shown in Figure 2.9a. The output of the sampler contains the frequency in the time domain represented by the impulse function in the frequency domain, as shown in Figure 2.9b. Thus the frequency response of the ZOH may be used to determine the amplitude spectrum of the data-hold output signal. The output signal components are shown in Figure 2.9c. Hence a signal of type shown in Figure 2.10 has the frequency spectrum of Figure 2.9c.

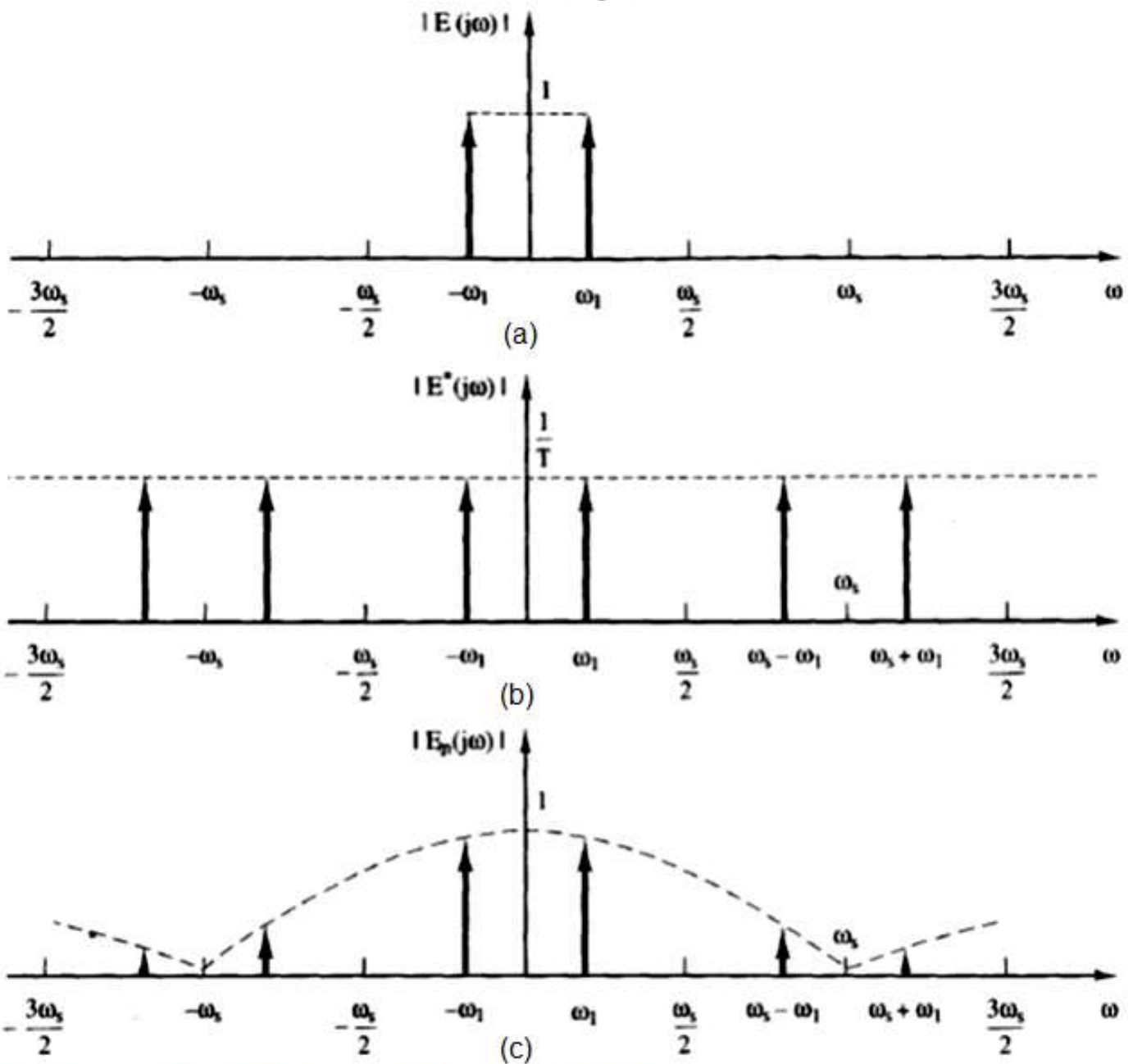


Figure 2.9 Sinusoidal response of the sampler and ZOH.

- (a) Input to the Ideal sampler.
- (b) Output signal of the Ideal sampler.
- (c) Output signal from the ZOH.

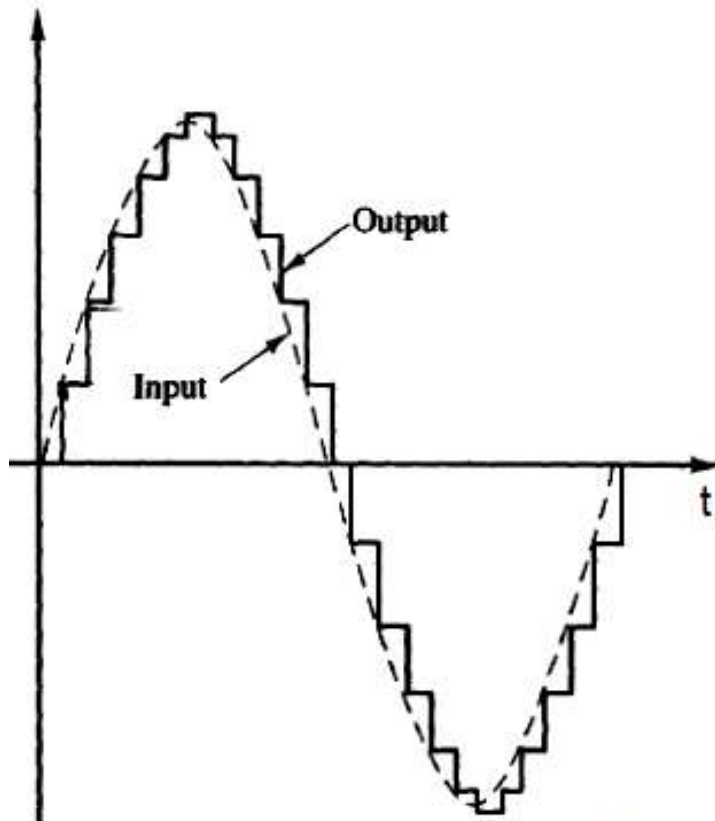


Figure 2.10 Output of the sampler - ZOH for sinusoidal input

## 2.7 First-Order Hold (FOH):

Using the current sample value and the previous one we might expect the below equation;

$$e'(nT) = \frac{e(nT) - e[(n-1)T]}{T}$$

This expression indicates that the extrapolated function within a given interval is a straight line and that its slope is determined by the values of the function at the sampling instants in the previous interval. Note that memory is required in the realization of this data hold, since  $e[(n-1)T]$  must be available at  $t=nt$ .



Therefore, by assuming the input as a **unit impulse** function and as shown in Figure 2.11 and the output formula will be

$$e_o(t) = u(t) + \frac{1}{T}t u(t) - 2u(t - T) - \frac{2}{T}(t - T)u(t - T) + u(t - 2T) + \frac{1}{T}(t - 2T)u(t - 2T)$$

therefore, the FOH transfer function will be;

$$G_{h1}(s) = \frac{E_o(s)}{E_i(s)} = \frac{1}{s} - \frac{2e^{-Ts}}{s} + \frac{e^{-2Ts}}{s} + \frac{1}{Ts^2}(1 - 2e^{-Ts} + e^{-2Ts})$$

Or;

$$G_{h1}(s) = \frac{1 + Ts}{T} \left[ \frac{1 - e^{-Ts}}{s} \right]^2$$

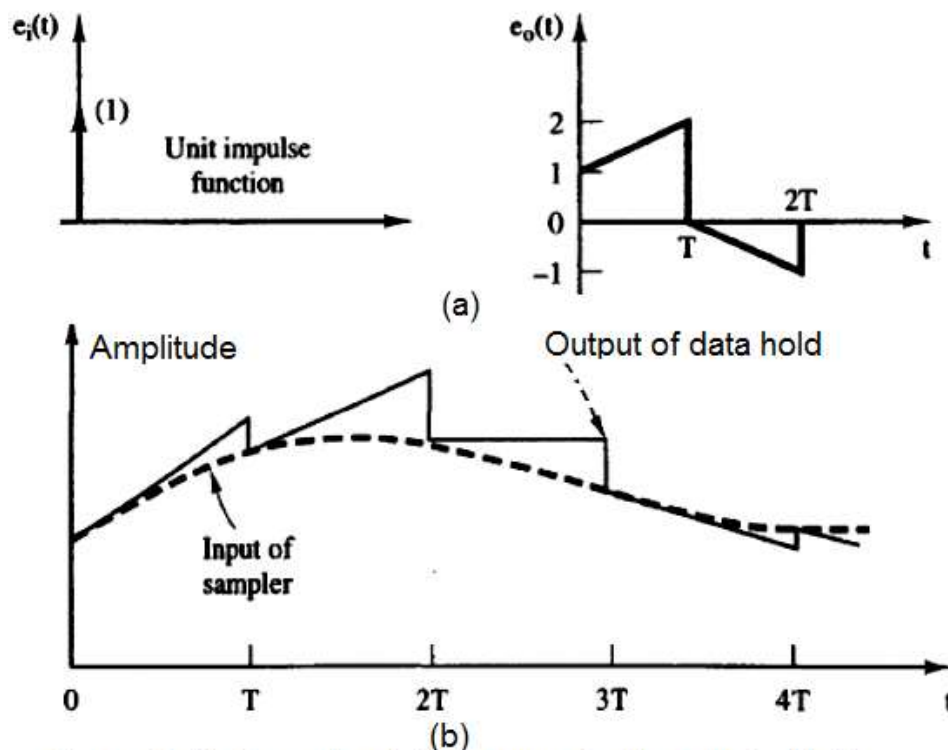


Figure 2.11 Example of the output of a First Order Hold



The Frequency response of the FOH is shown below and in Figure 2.12.

$$G_{h1}(j\omega) = \frac{1 + j\omega T}{T} \left[ \frac{1 - e^{-j\omega T}}{j\omega} \right]^2$$

$$|G_{h1}(j\omega)| = T \sqrt{1 + \frac{4\pi^2 \omega^2}{\omega_s^2} \left[ \frac{\sin(\pi\omega/\omega_s)}{\pi\omega/\omega_s} \right]^2}$$

$$\angle G_{h1}(j\omega) = \tan^{-1} \left( \frac{2\pi\omega}{\omega_s} \right) - \frac{2\pi\omega}{\omega_s}$$

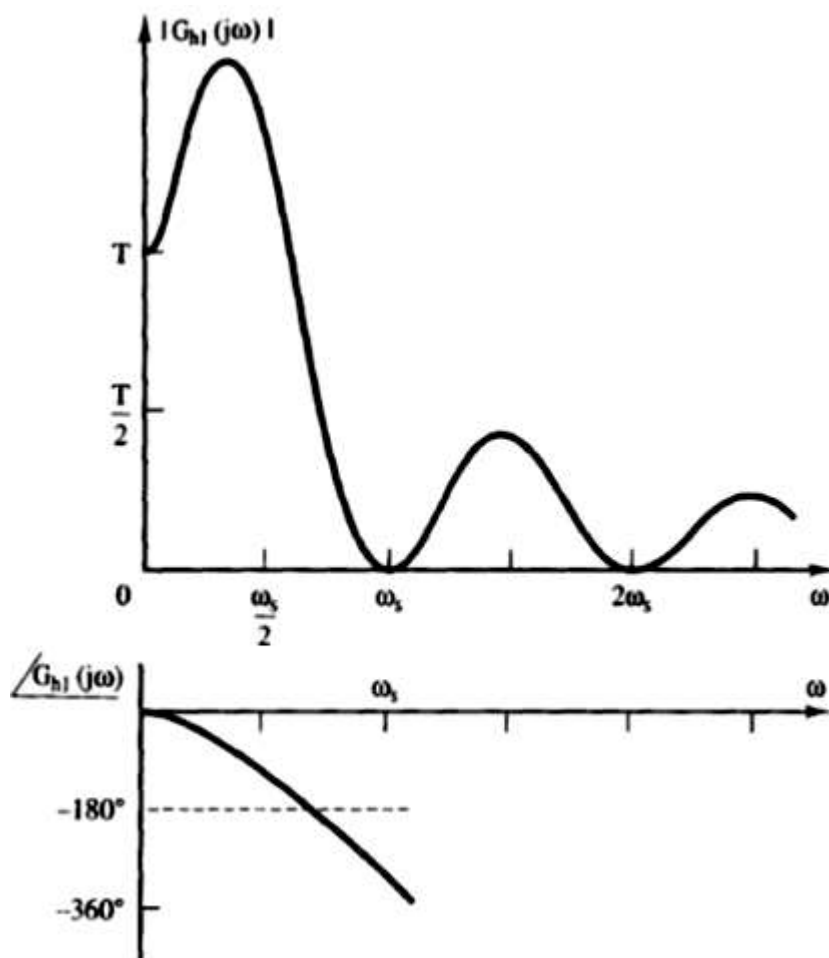
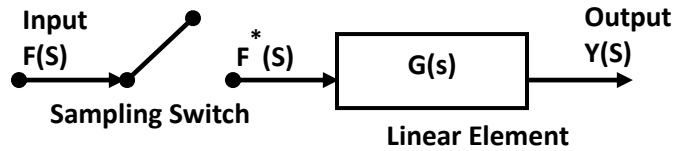


Figure 2.12 Frequency response of the FOH



## 2.8 Blocked Diagrams of an open loop sampled Data control system:

❖ Suppose we have the below system:

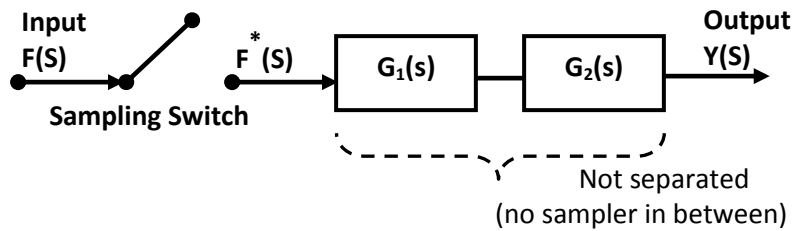


$$Y(S)=F^*(S) G(S)$$

Converting to Z-Domain;

$Y(z)=F(z) G(z)$  , where  $G(z)$  called the pulse transfer function of the system.

❖ For the below Sampler;



$$Y(S)=F^*(S) G_1(S) G_2(S)$$

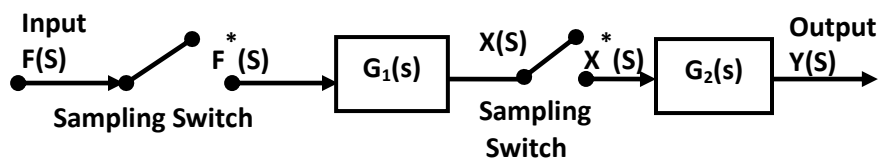
And

$$Y(z)=F(z) G_1G_2(z)$$

❖ For the below double Sampler ;

$$X(S)= G_1(S) F^*(S)$$

$$Y(S)= G_2(S) X^*(S)$$



And since;  $X^*(S)= G_1^*(S) F^*(S)$

Then  $Y(S)= G_2(S) G_1^*(S) F^*(S)$  or  $y(S)= G_1^*(S) G_2(S) F^*(S)$

Thus  $y^*(S)= G_1^*(S) G_2^*(S) F^*(S)$





By letting  $z=e^{Ts}$  leads to;

$$Y(Z)=G_1(z) G2(z) F(z)$$

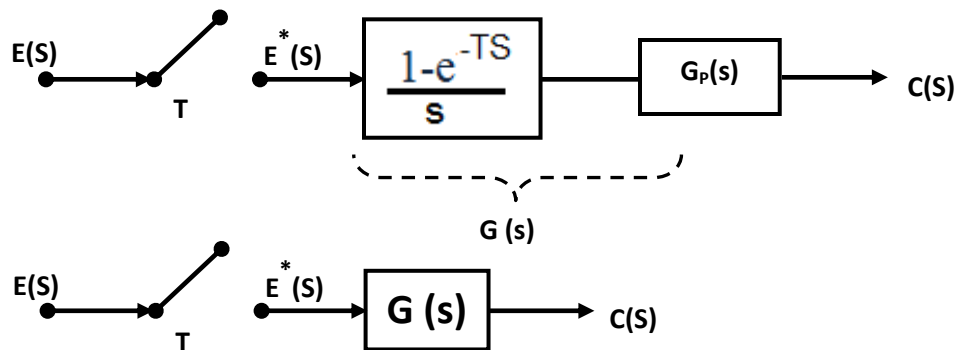
This equation implies that the z-transform of the output is equal to the z-transform of the input multiplied by the pulse transfer function of systems  $G1(s)$  and  $G2(s)$ .

Where  $G(z)$  is the overall Pulse Transfer Function (PTF) and the system defined as;

$$G(z) = G1(z)G2(z) = \frac{Y(z)}{F(z)}$$

Consider the open-loop system shown in Figure 2.13, where  $G_p(s)$  is the plant transfer function. We denote the product of the plant transfer function and zero-order hold transfer function as  $G(s)$ , which may be written as

$$G(s) = G_{h0}(s)G_p(s) = \frac{1 - e^{-Ts}}{s} G_p(s)$$



**Figure 2.13 Open Loop Sampled Data System**

Now deriving the discrete output from the above figure as;

$$C(s) = G(s)E^*(s) \quad \text{and then} \quad C^*(s) = [G(s)E^*(s)]^*$$

Therefore,

$$C(z) = E(z)G(z)$$



The derivation above is completely general. Thus, given any function that can be expressed as;

$$A(s) = B(s)F^*(s)$$

Where  $F^*(s)$  is the sampled version of  $F(s)$ . Then, from the preceding development,

$$A^*(s) = B^*(s)F^*(s)$$

Where  $B(s)$  is a function of  $s$  (continuous) and  $F^*(s)$  is function of  $e^{Ts}$  (discrete), thus;

$$F(z) = F^*(s) \Big|_{e^{Ts}=z}$$

#### Example 2.4:

Suppose that we wish to find the z-transform of:

$$A(s) = \frac{1 - e^{-Ts}}{s(s+1)} = (1 - e^{-Ts}) \frac{1}{s(s+1)}$$

Accordingly let;

$$B(s) = \frac{1}{s(s+1)}$$

and since each  $z=e^{Ts}$ , then the term  $(1 - e^{-Ts})$  is always a discrete term and may be written as;

$$F^*(s) = 1 - e^{-Ts} \rightarrow F(z) = 1 - z^{-1} = \frac{z-1}{z}$$

Now using residue method;

$$B(z) = z \left\{ \frac{1}{s(s+1)} \right\} = \left[ \frac{1}{(s+1)} \frac{z}{z - e^{Ts}} \right] \Big|_{s=0} + \left[ \frac{1}{s} \frac{z}{z - e^{Ts}} \right] \Big|_{s=-1}$$

$$B(z) = \frac{z}{z-1} - \frac{z}{z-e^{-T}} = \frac{z(1 - e^{-T})}{(z-1)(z - e^{-T})}$$



And then,

$$A(z) = \frac{z-1}{z} \left[ \frac{z(1-e^{-T})}{(z-1)(z-e^{-T})} \right] = \frac{1-e^{-T}}{(z-e^{-T})}$$

### Example 2.5:

Given the system shown in Figure 2.14, with input  $e(t)$  as a unit step function, determine the output function  $C(z)$ .

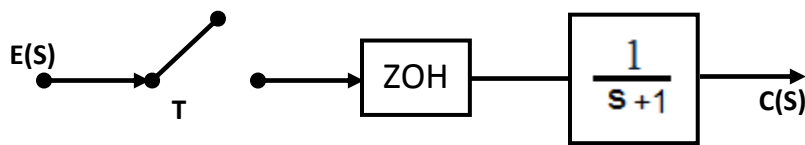


Figure 2.14 Open Loop Sampled Data System Example

Now;

$$C(s) = G_{h0}(s)G_P(s)E^*(s)$$

Where  $G_{h0}(s)$  is the ZOH T.F. and  $G_P(s)$  is the  $1/(s+1)$ . Then;

$$C^*(s) = G_{h0}G_P^*(s)E^*(s)$$

and;

$$C(z) = \overline{G_{h0}G_P}(z)E(z)$$

Now since  $e(t)$  is known as the unit step input then  $E(z) = \frac{z}{z-1}$

Therefore we have to find;

$$\begin{aligned} \overline{G_{h0}G_P}(z) &= \mathcal{Z}\{G_{h0}(s)G_P(s)\} = \mathcal{Z}\left\{\frac{1-e^{-Ts}}{s} \frac{1}{(s+1)}\right\} \\ \overline{G_{h0}G_P}(z) &= \frac{z-1}{z} \mathcal{Z}\left\{\frac{1}{s(s+1)}\right\} = \frac{z-1}{z} \left[ \frac{z}{z-1} - \frac{z}{z-e^{-T}} \right] \end{aligned}$$

then,

$$C(z) = \frac{z-1}{z} \left[ \frac{z}{z-1} - \frac{z}{z-e^{-T}} \right] \frac{z}{z-1} = \frac{z}{z-1} - \frac{z}{z-e^{-T}}$$



And the inverse Z-transform of  $C(z)$  is;

$$C(nT) = u(nT) - u(nT)e^{-nT} = 1 - e^{-nT}$$

This response is shown in Figure 2.15 for  $T = 0.5$  sec, by substituting  $n=0,1,2,3,4,\dots$

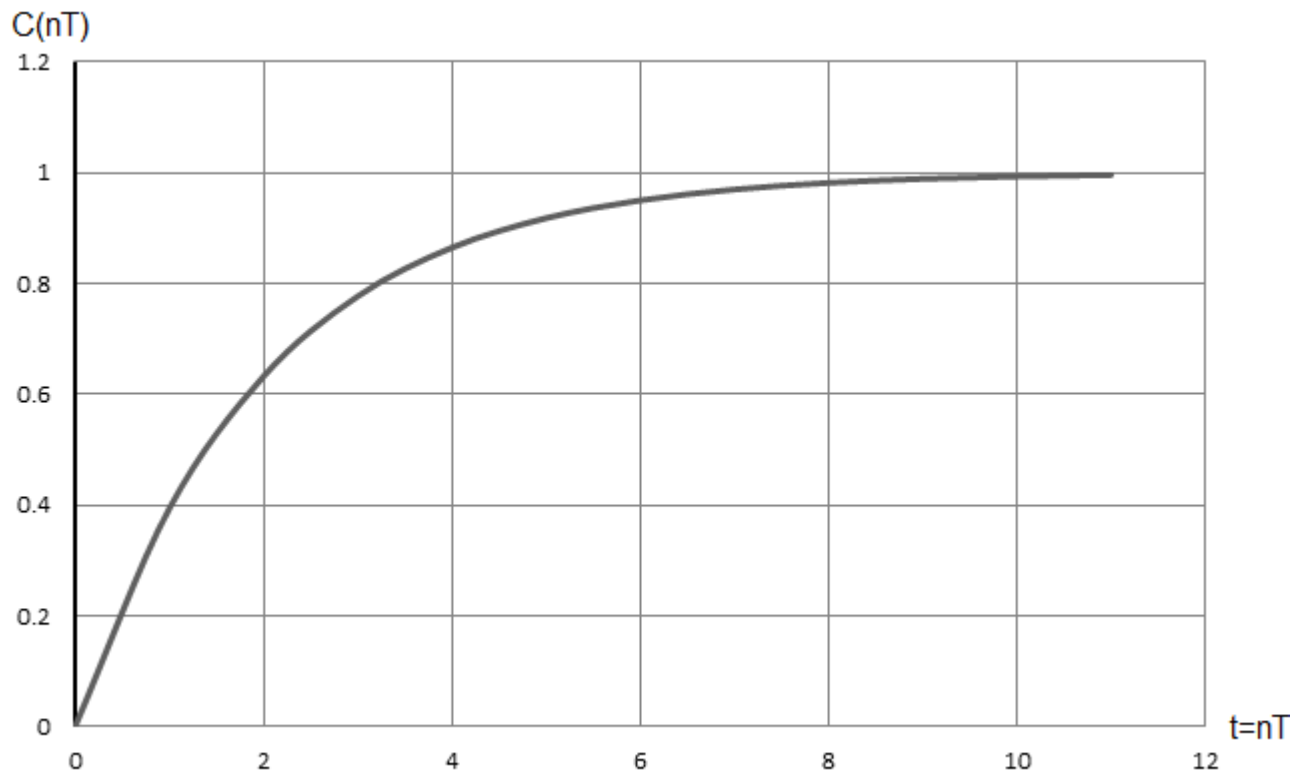


Figure 2.15 Sampled output function  $C(nT)$

## 2.9 Closed - loop sampled Data control system:

Consider the system shown in Figure 2.16. First we write;

$$C(s) = G(s)E^*(s)$$

and

$$E(s) = R(s) - H(s)C(s)$$

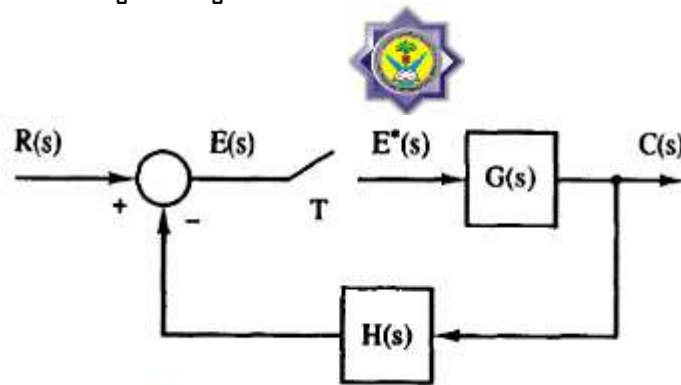


Figure 2.16 Closed - Loop Samled Data System

Substituting  $C(s)$  equation into  $E(s)$ ;

$$E(s) = R(s) - G(s)H(s)E^*(s)$$

By Starring both sides of this equation, get;

$$E^*(s) = R^*(s) - GH^*(s)E^*(s)$$

$G(s)H(s)$  term was combined into  $GH^*(s)$  as both  $G(s)$  and  $H(s)$  are continuous. Now Solving for  $E^*(s)$ ;

$$E^*(s) = \frac{R^*(s)}{1 + GH^*(s)}$$

Substituting this result in the equation of  $C(s)$ ;

$$C(s) = \frac{G(s)R^*(s)}{1 + GH^*(s)}$$

Staring;

$$C^*(s) = \frac{G^*(s)R^*(s)}{1 + GH^*(s)}$$

And Finally the z-transform of the output is;

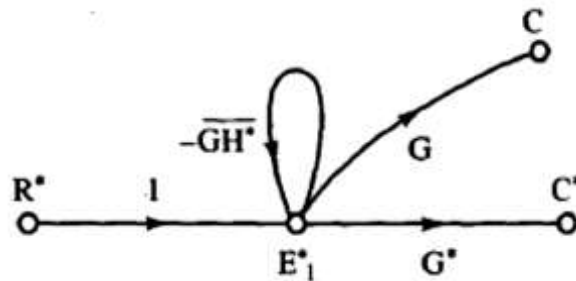
$$\boxed{C(z) = \frac{G(z)R(z)}{1 + \overline{GH}(z)}}$$



Where  $\overline{GH}(z) = Z[G(s)H(s)]$ . Now the flow graph of the original system is also shown below



While the sampled signal flow graph for the output result is ;



**Example 2.6:**

For the sampled data system shown in Figure 2.17, derive an expression for C(z);

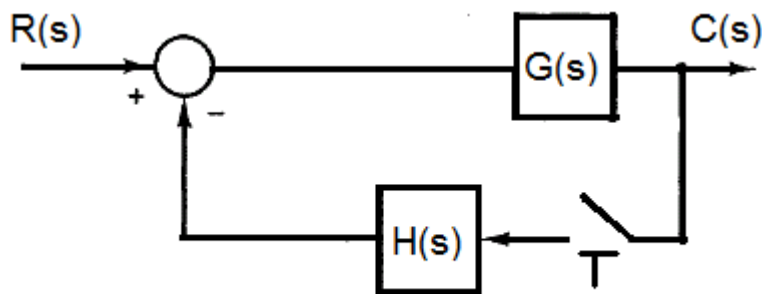


Figure 2.17 Sampled Data System

The system may be analyzed as follows;

$$C(s) = G(s)E(s)$$

also;



$$E(s) = R(s) - H(s)C^*(s)$$

Substituting this result of  $E(s)$  into  $C(s)$  equation, results in;

$$C(s) = G(s)R(s) - G(s)H(s)C^*(s)$$

Staring both sides;

$$C^*(s) = GR^*(s) - GH^*(s)C^*(s)$$

Note that  $G(s)$  and  $R(s)$  were combined together and the same for  $G(s)$  and  $H(s)$ .

Solving for  $C^*(s)$  and then for  $C(z)$ ;

$$C^*(s) = \frac{GR^*(s)}{1 + GH^*(s)}$$

and finally

$$C(z) = \frac{\overline{GR}(z)}{1 + \overline{GH}(z)}$$

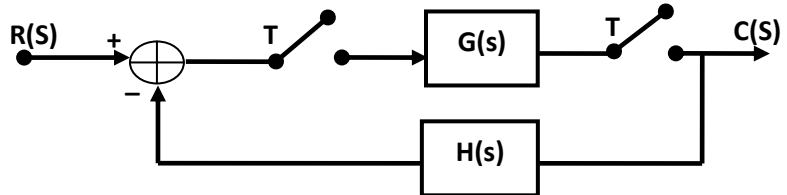
Since the input  $R(s)$  was combined with  $G(s)$ , therefore, no transfer function can be derived. This is because the input was not sampled before being applied to the analog part of the system

### Exercises:

**Below Various of sampled data control systems;**

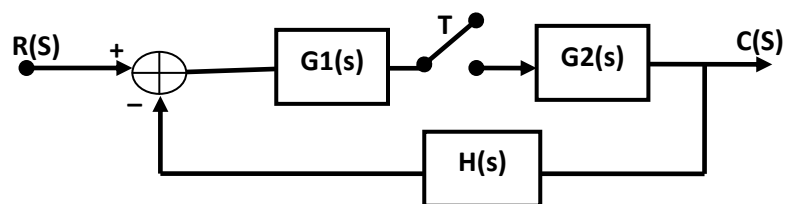
1-

$$C(z) = \frac{G(z)R(z)}{1 + G(z)H(z)}$$



2-

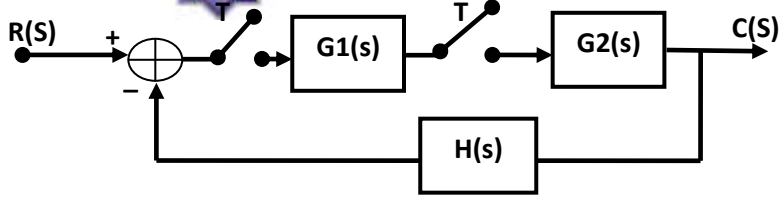
$$C(z) = \frac{G_2(z)\overline{RG_1}(z)}{1 + \overline{G_1G_2H}(z)}$$





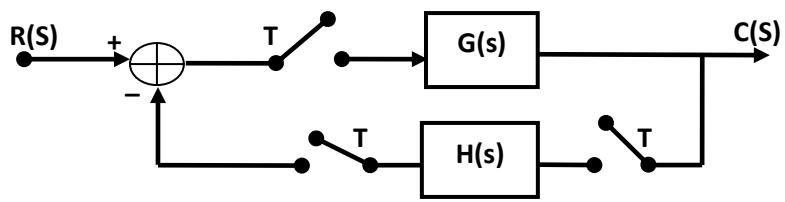
3-

$$C(z) = \frac{G_1(z)G_2(z)}{1 + G_1(z)G_2H(z)} R(z)$$



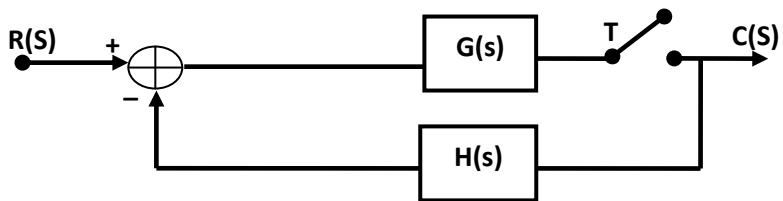
4-

$$C(z) = \frac{G(z)}{1 + G(z)H(z)} R(z)$$



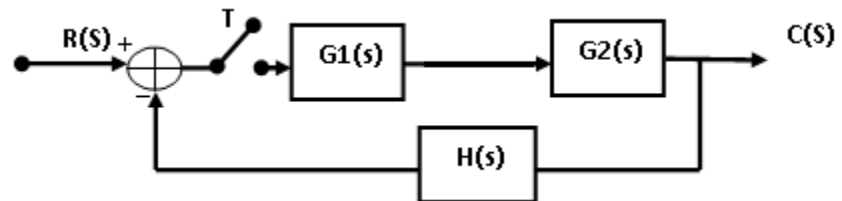
5-

$$C(z) = \frac{\overline{RG}(z)}{1 + \overline{HG}(z)}$$



6-

$$C(z) = \frac{\overline{G_1G_2}(z)}{1 + \overline{G_1G_2}H(z)} R(z)$$



**Example 2.7:**

Consider the digital control system shown in Figure 2.18a. This system contains a digital block (discrete block) together with continuous blocks. Figure 2.18b shows a corresponding system model.

Now the derivation of the output C(z) is as follows;

$$C(s) = E^*(s)D^*(s)G_{ho}(s)G_p(s)$$

D(z) was substituted here as a star because the block is a digital one (discrete). Also ;





$$E(s) = R(s) - C(s)H(s)$$

Now before starting both sides, first we have to substitute the equation of C(s) into the equation of E(s) as follows;

$$E(s) = R(s) - E^*(s)D^*(s)G_{ho}(s)G_p(s)H(s)$$

Starting both sides:

$$E^*(s) = R^*(s) - E^*(s)D^*(s)G_{ho}G_pH^*(s)$$

Note that only the continuous function may be combined together. Solving for E\*(s);

$$E^*(s) = \frac{R^*(s)}{1 + D^*(s)G_{ho}G_pH^*(s)}$$

Substituting this equation into C(s) equation:

$$C(s) = \frac{R^*(s)D^*(s)G_{ho}(s)G_p(s)}{1 + D^*(s)G_{ho}G_pH^*(s)}$$

And finally;

$$C(z) = \frac{R(sz)D(z)\overline{G_{ho}}\overline{G_p}(z)}{1 + D(z)\overline{G_{ho}}\overline{G_p}H(s)}$$

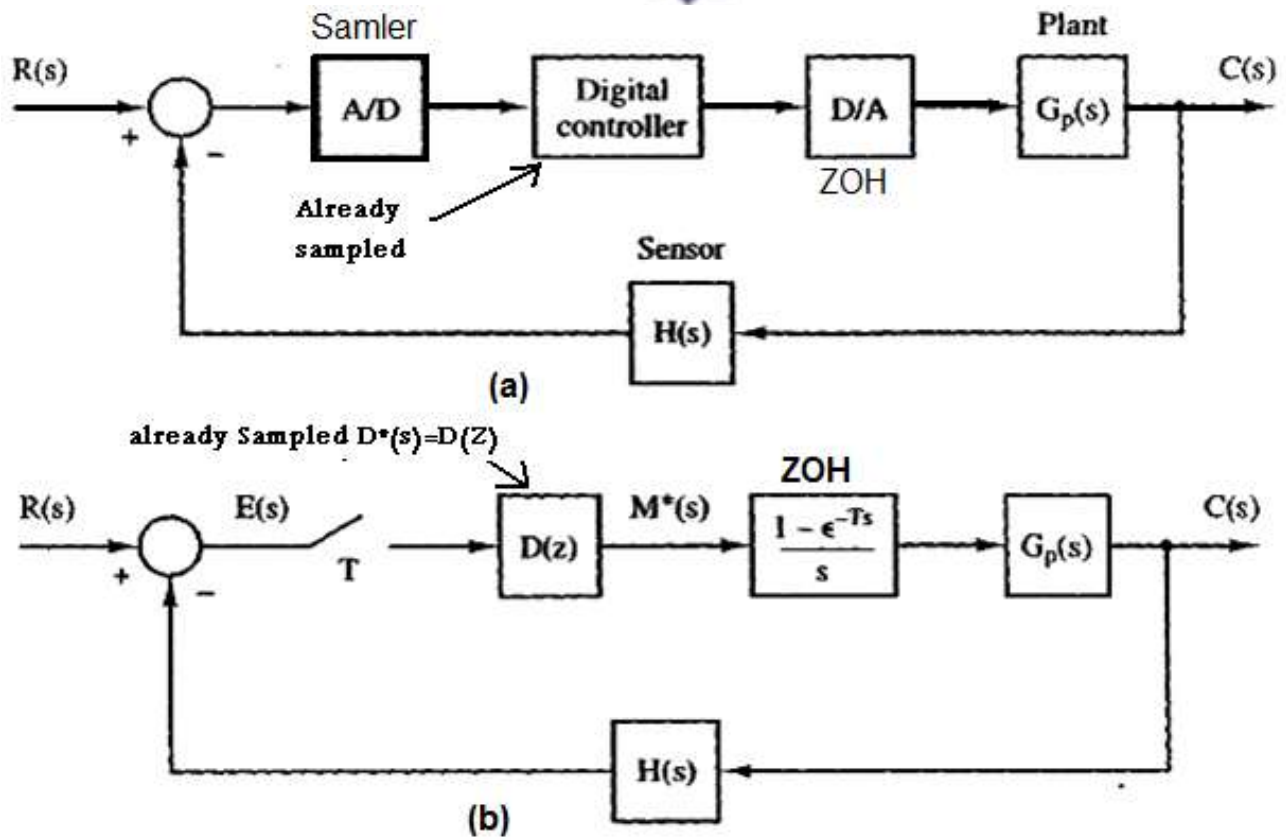


Figure 2.18 Closed loop digital control system

**Example 2.8:**

For the system shown in Figure 2.19, Find  $C(z)$ .

The flow graph of this system is shown in Figure 2.20.

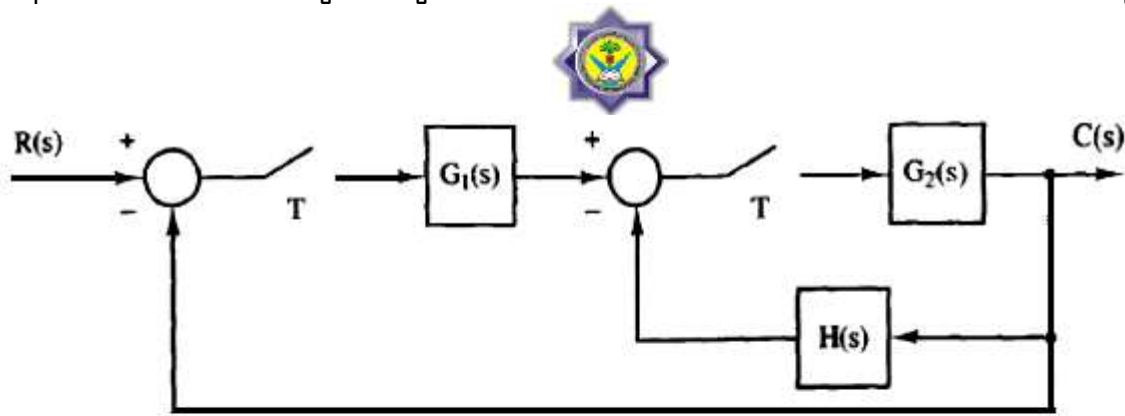


Figure 2.19 Example of sampled data system

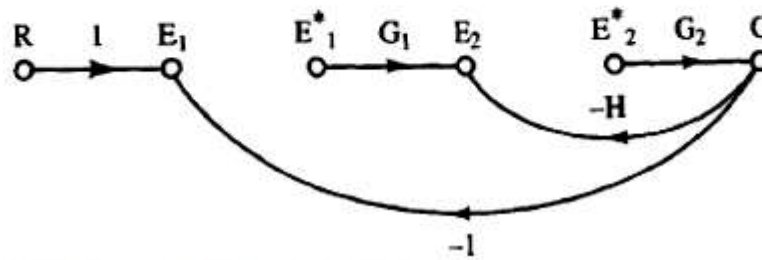


Figure 2.20 Original flow graph

$$C(s) = G_2(s)E_2^*(s)$$

$$E_2(s) = E_1^*(s)G_1(s) - H(s)C(s) = E_1^*(s)G_1(s) - H(s)G_2(s)E_2^*(s)$$

and ;

$$E_1(s) = R(s) - C(s) = R(s) - G_2(s)E_2^*(s)$$

Starring both sides;

$$E_1^*(s) = R^*(s) - G_2^*(s)E_2^*(s)$$

Substituting  $E_2(s)$  equation;

$$E_2(s) = [R^*(s) - G_2^*(s)E_2^*(s)]G_1(s) - H(s)G_2(s)E_2^*(s)$$

$$E_2(s) = R^*(s)G_1(s) - G_1(s)G_2^*(s)E_2^*(s) - H(s)G_2(s)E_2^*(s)$$

Starring both sides;

$$E_2^*(s) = R^*(s)G_1^*(s) - G_1^*(s)G_2^*(s)E_2^*(s) - HG_2^*(s)E_2^*(s)$$



then;

$$E_2^*(s) = \frac{R^*(s)G_1^*(s)}{1 + G_1^*(s)G_2^*(s) + HG_2^*(s)}$$

Substituting this result in the equation of C(s);

$$C(s) = \frac{R^*(s)G_1^*(s)G_2(s)}{1 + G_1^*(s)G_2^*(s) + HG_2^*(s)}$$

Resulting in ;

$$C(z) = \frac{R(z)G_1(z)G_2(z)}{1 + G_1(z)G_2(z) + \overline{HG_2}(z)}$$

The sampled flow graph can be found as shown in Figure 2.21

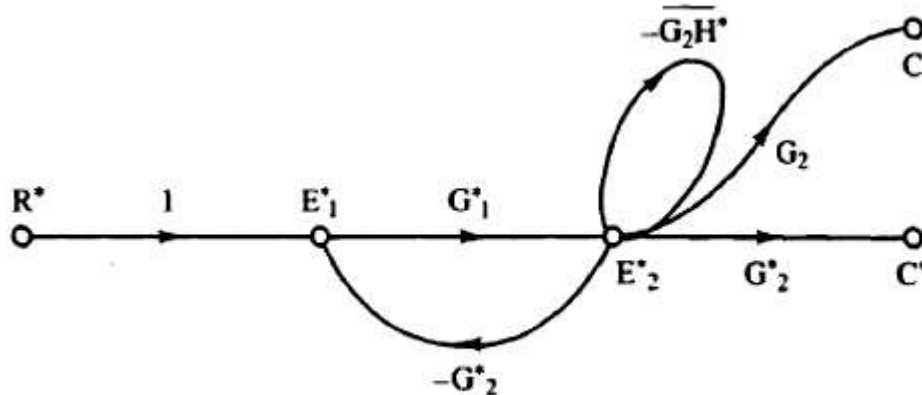
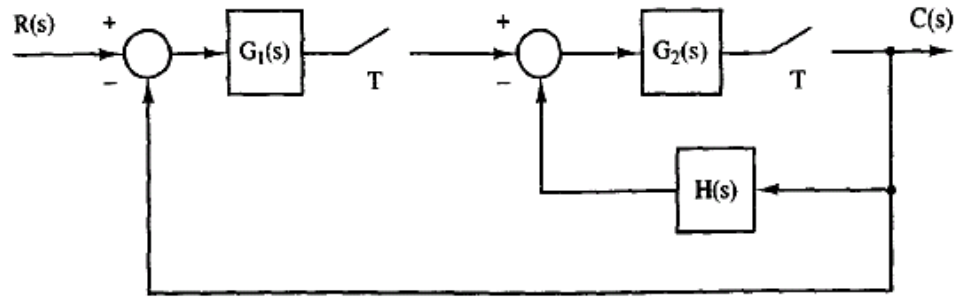


Figure 2.21 Sampled flow graph



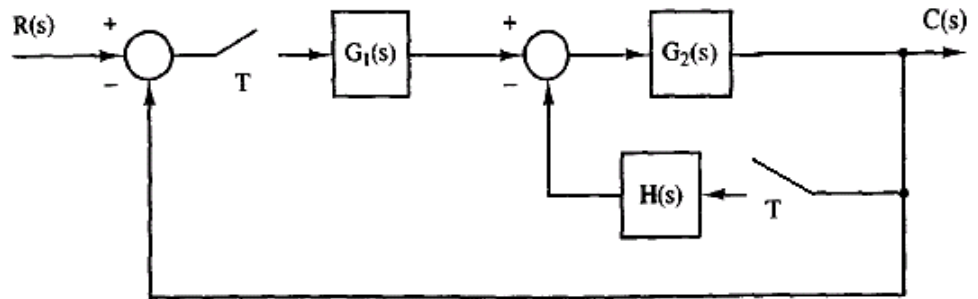
**Exercises :** Derive expressions for C(z) for below sampled data control systems.

1-



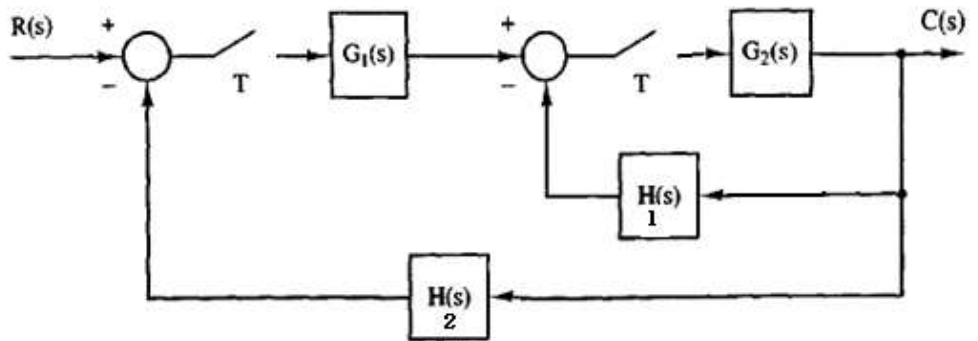
Ans:  $C(z) = \frac{\overline{RG_1(z)G_2(z)}}{1 + \overline{G_1(z)G_2(z)} + \overline{HG_2(z)}}$

2-



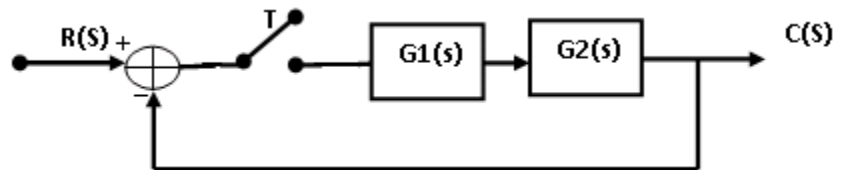
Ans:  $C(z) = \frac{\overline{G_1G_2(z)}}{1 + \overline{G_1G_2(z)} + \overline{HG_2(z)}} R(z)$

3-



Ans:  $C(z) = \frac{\overline{G_1(z)G_2(z)}}{1 + \overline{G_2H_2(z)G_1(z)} + \overline{G_2H_1(z)}}$  R(z)

4-



Ans:  $C(z) = \frac{\overline{G_1G_2(z)}}{1 + \overline{G_1G_2(z)}}$  R(z)

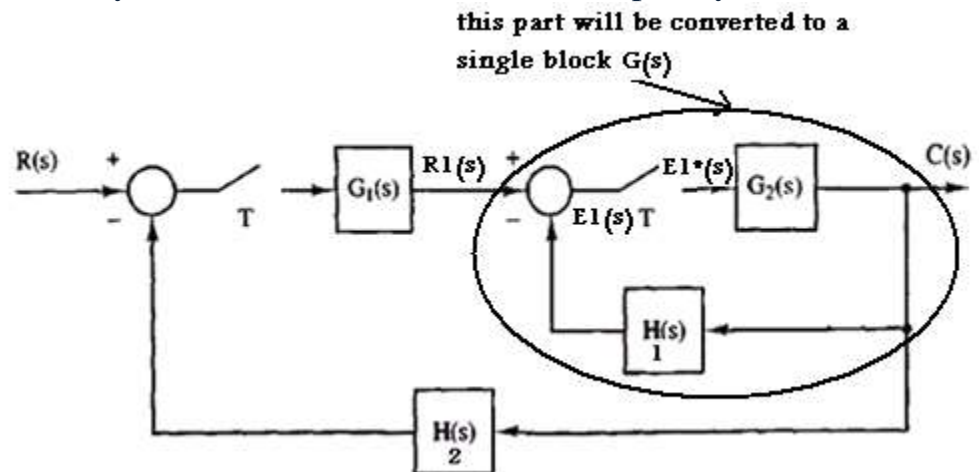


## 2.10 Data Sampled Control System reduction:

Systems might be reduced to another simpler block diagrams by system reduction as shown through the below example.

### Example 2.9:

Reduce the data sampled control system shown below to another simpler system.



Finding the transfer function of the part under the circle  $G(s)$ :

$$E_1(s) = R_1(s) - C(s) H_1(s) = R_1(s) - E_1^*(s) G_2(s) H_1(s)$$

Therefore after starring both sides, we get;

$$E_1^*(s) = R_1^*(s) - E_1^*(s) \overline{G_2 H_1}^*(s)$$

$$\text{Then } E_1^*(s) = \frac{R_1^*(s)}{1 + \overline{G_2 H_1}^*(s)}$$

$$\text{And since } C^*(s) = E_1^*(s) G_2^*(s)$$

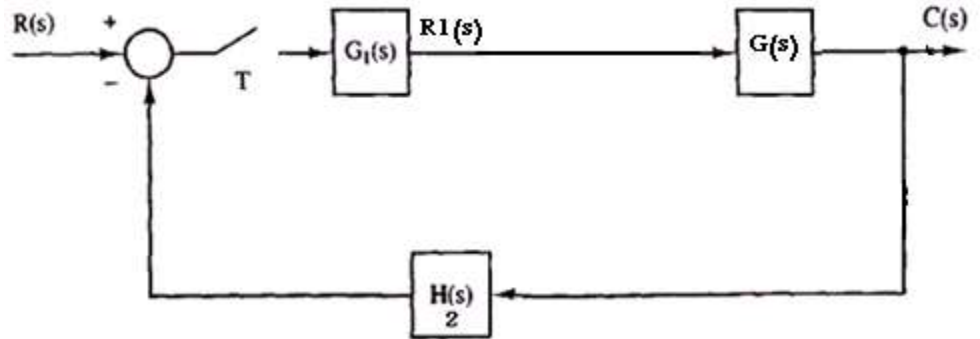
$$\text{Then } C^*(s) = \frac{R_1^*(s) G_2^*(s)}{1 + \overline{G_2 H_1}^*(s)}$$



or 
$$\frac{C^*(s)}{R_1^*(s)} = G^*(s) = \frac{G_2^*(s)}{1 + \overline{G_2 H_1}^*(s)}$$

Also 
$$C(z) = \frac{R_1(z) G_2(z)}{1 + \overline{G_2 H_1}(z)}$$

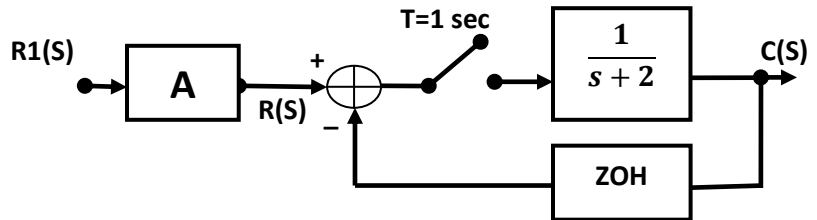
Then the system will be reduced to the one shown below.



**Example 2.10:**

Derive and find C(Z) for t(t) as a unit step function and for each case of block  $A(S) = \frac{R(S)}{R1(S)}$ :

- 1- A is a Differentiator.
- 2- A is an Integrator.



**Solution:**

Let  $G(S) = \frac{1}{(s+2)}$  and  $H(S)=ZOH$

Then

$$C(Z) = \frac{G(Z)}{1 + \overline{HG}(Z)} R(Z) \quad \text{?????}$$

Now G(s) has one residue only;

$$G(Z) = \frac{Z}{Z - e^{-2}} \quad \text{???????$$

and



$$\overline{HG}(Z) = \frac{1}{2} - \frac{Z-1}{2(Z-e^{-2})} = \text{??????????}$$

Then ;

$$C(Z) = \frac{\frac{Z}{Z-e^{-2}}}{1 + \frac{1}{2} \frac{(Z-1)}{2(Z-e^{-2})}} R(Z) = \frac{Z}{\frac{3}{2}Z - \frac{3}{2}e^{-2} - \frac{1}{2}Z + \frac{1}{2}} R(Z) \tag{1}$$

Now;

1- For A acting as a differentiator:

$$R(S) = S R_1(S) = s \frac{1}{S} = 1$$

Then R(Z)=1. Substitute in equation (1).

2- For A as an integrator;

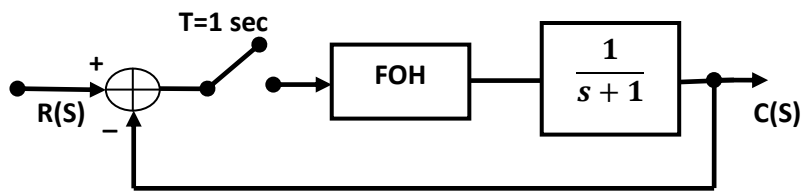
$$R(S) = \frac{1}{S} R_1(S) = \frac{1}{S^2}$$

Then  $R(Z) = \frac{Z}{(Z-1)^2}$

Substitute in equation (1).

**H.W:**

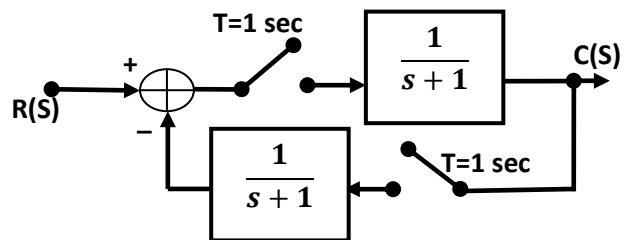
Find C(Z) / R(Z)



**H.W:**

For the sampled data system below ; find C(z) for input as;

1- impulse and 2- unit step







## 2.11 Direct Discrete-Time Controller Design / Fixed Structure Controllers

### ❖ Analog PID Controller

The output equation of the analog PID controller is;

$$u(t) = K \left[ \underbrace{e(t)}_{\text{Proportional}} + \underbrace{\frac{1}{T_i} \int_0^t e(\tau) d\tau}_{\text{Integral}} + \underbrace{T_d \frac{de(t)}{dt}}_{\text{Derivative}} \right]$$

where  $K$ ,  $T_i$ , and  $T_d$  are the proportional gain, the integral (or reset) time, and the derivative (or rate) time respectively. In practice, derivative part approximated by including a pole “far” from the  $j\omega$ -axis as shown below

$$U(s) = K \left[ 1 + \frac{1}{T_i s} + \frac{T_d s}{1 + \frac{T_d}{N} s} \right] E(s)$$

a “small”  $T_d$  and a “large”  $N$  (3 to 10) produce a “good” approximation

### ❖ Discrete PID Controller

It can be obtained by “discretizing” the analog PID controller.

**DIFFERENTIATION:** Two-point difference form

$$\frac{de(t)}{dt} = \lim_{T \rightarrow 0} \frac{e(t) - e(t-T)}{T} \approx \frac{e(t) - e(t-T)}{T}$$

$$\Rightarrow \left. \frac{de(t)}{dt} \right|_{t=kT} \approx \frac{e(kT) - e((k-1)T)}{T}$$

Using the two-point difference form for differentiation and trapezoidal approximation for integration leads to the pulse transfer function for a digital PID controller.



$$G_D(z) = \frac{U(z)}{E(z)} = K_P + \frac{K_I}{1 - z^{-1}} + K_D (1 - z^{-1})$$

where

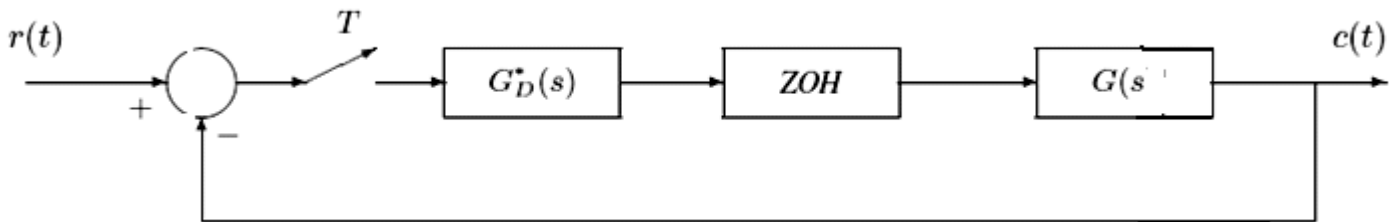
$$K_P = K - \frac{KT}{2T_i} = K - \frac{K_I}{2} = \text{proportional gain}$$

$$K_I = \frac{KT}{T_i} = \text{integral gain}$$

$$K_D = \frac{KT_d}{T} = \text{derivative gain}$$

### Example 2.11:

Consider the control system with a digital PID controller shown in the below Figure.



Suppose the sampling time is  $T = 1$  sec and the plant is

$$G_P(s) = \frac{1}{s(s+1)}$$

Determine the pulse transfer function  $G_D(z)$  of the PID controller for

$$K_P = 1; K_I = 0.2; K_D = 0.2,$$

and hence, the closed loop pulse transfer function.



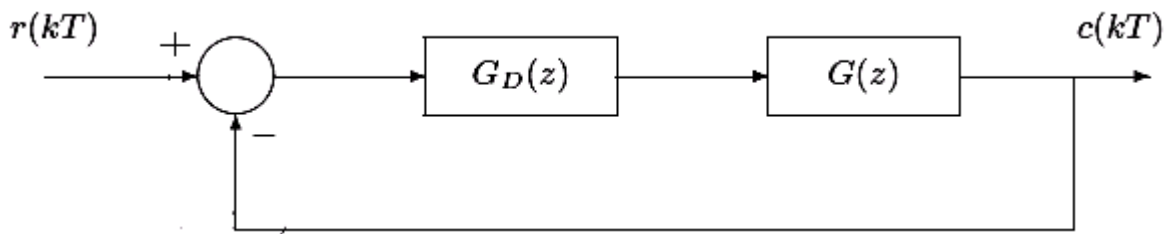
**Solution:**

$$G(z) = (1 - z^{-1}) \mathcal{Z} \left\{ \frac{G_P(s)}{s} \right\}$$

$$= \frac{0.3679 Z^{-1} + 0.2642 Z^{-2}}{(1 - 0.3679 Z^{-1})(1 - Z^{-1})}$$

and  $G_D(z) = \frac{1.4 - 1.4z^{-1} + 0.2z^{-2}}{1 - z^{-1}}$

Now the block diagram can be redrawn as



$$\frac{C(z)}{R(z)} = \frac{G_D(z)G(z)}{1 + G_D(z)G(z)}$$

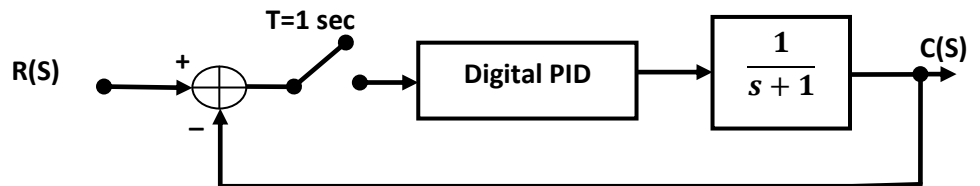
$$= \frac{0.5151z^{-1} - 0.1452z^{-2} - 0.2963z^{-3} + 0.0528z^{-4}}{1 - 1.8528z^{-1} + 1.5906z^{-2} - 0.6642z^{-3} + 0.0528z^{-4}}$$

**Example 2.12:**

Find C(Z) for the system below:

$K_P=1;$

$K_I=K_D=0.2;$



**Solution:**

Let  $G_1(Z)$ =the digital PID controller transfer function in Z-domain.  
and  $G_2(S)=1/(S+1)$

then;

$$C(Z) = \frac{G_1(Z)G_2(Z)}{1 + G_1(Z)G_2(Z)} R(Z)$$

also;

$$G_1(Z) = 1 + \frac{0.2 Z}{Z - 1} + \frac{0.2 (Z - 1)}{Z}$$

and

$$G_2(Z) = \frac{Z}{Z - 1} - \frac{Z}{Z - e^{-1}}$$

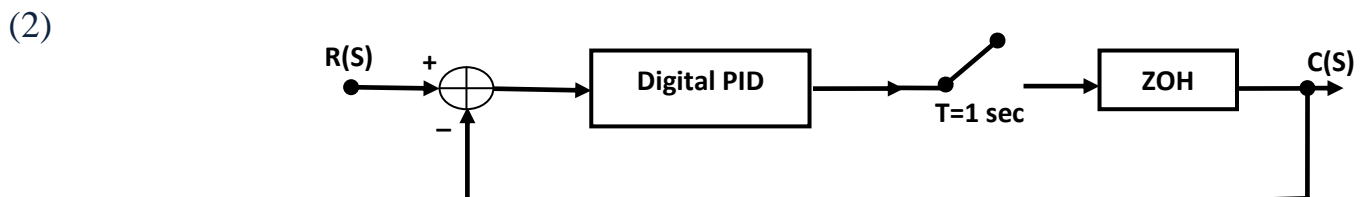
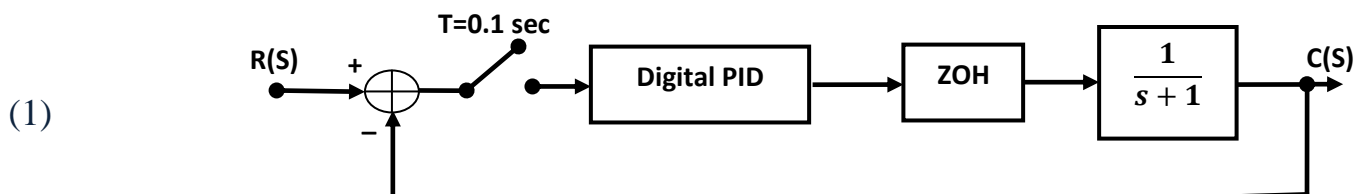
Substitute both  $G_1$  and  $G_2$  in equation of  $C(Z)$ .

**Exercises:**

Find  $C(Z)$  for the below discrete system;

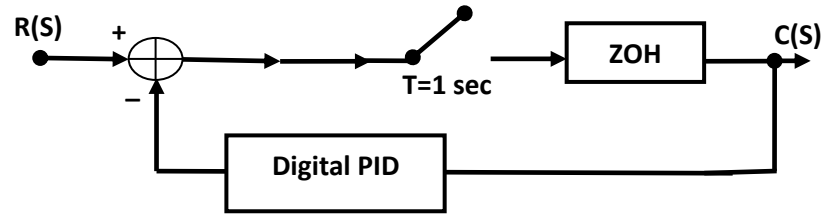
PID constants;

$K_P=1; K_I=K_D=0.4;$





(3)



(4)

