System Time Response Characteristics:

3.1 Introduction

After the discrete system was introduced, the time response of it will be investigated. Next, regions in the s-plane are mapped into regions in the z-plane. Then by using the correlation between regions in the two planes, the effect of the closed-loop z-plane poles on the system transient response is discussed. Next, the effects of the system transfer characteristics on the steady state system error are considered.

Now, the time response of discrete-time systems is introduced via the following examples.

Example 3.1:

Find the unit-step response for the first order system shown in Fig.3.1 below with sampling time T=0.1 sec. Since the system of a temperature control system is often modeled as a first order system, then this example might be considered as a model of temperature control system.

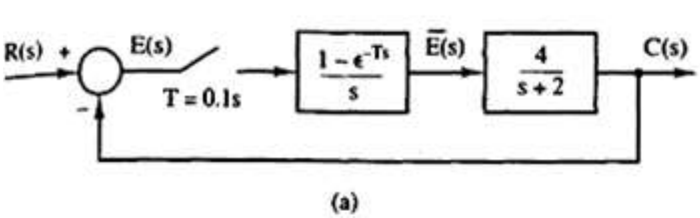
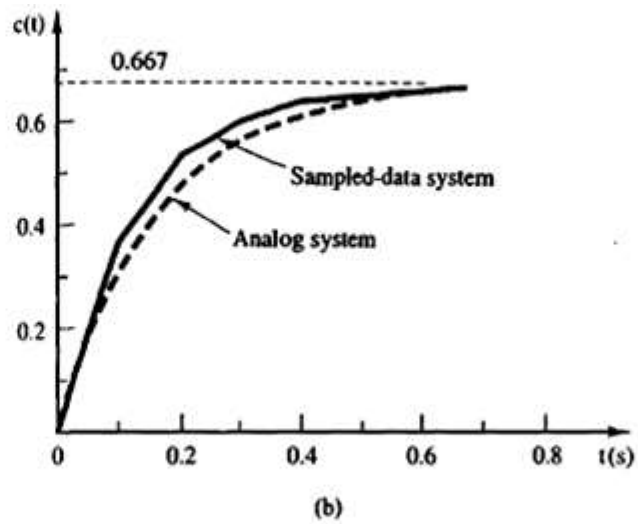


Fig 3.1 System and response



Now, using simple analysis;

$$C(z) = \frac{G(z)}{1 + G(z)} R(z)$$

Where G(z) is ;

$$G(z) = \mathcal{Z}\left[\frac{1-\epsilon^{-Ts}}{s}\frac{4}{s+2}\right] = \frac{z-1}{z}\mathcal{Z}\left[\frac{4}{s(s+2)}\right] = \frac{z-1}{z}\frac{2(1-\epsilon^{-2T})z}{(z-1)(z-\epsilon^{-2T})}$$

using $T = 0.1$ s , then ;

$$G(z) = \frac{0.3625}{z-0.8187},$$

Therefore, the closed system transfer function T(z) is;

$$T(z) = \frac{C(z)}{R(z)} = \frac{G(z)}{1+G(z)} = \frac{0.3625}{z-0.4562}$$

For a unit step function input R(t); $R(z) = \mathcal{Z}[1/s] = z/(z-1),$

Then ; $$C(z) = \frac{0.3625z}{(z-1)(z-0.4562)} = \frac{0.667z}{z-1} + \frac{-0.667z}{z-0.4562}$$

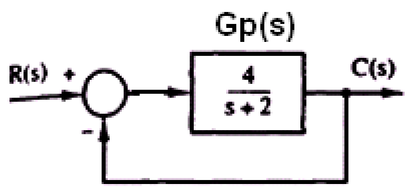
and by taking the inverse z-transform (to get c(kt));

$$c(kT) = 0.667[1-(0.4562)^k] \ u(kT)$$

By taking kT from 0.1 to 0.8 we may draw the system discrete time response using below table (response shown in Fig.1.b). From the table and the figure as well it is seen that the discrete response reaches a steady state of value 0.667.

kT	$c(kT)$	$c_a(t)$
0	0	0
0.1	0.363	0.300
0.2	0.528	0.466
0.3	0.603	0.557
0.4	0.639	0.606
0.5	0.654	0.634
0.6	0.661	0.648
⋮		
1.0	0.666	0.665

Now; to show the effect of sampling on the system response, we will remove the sampler and zero-order hold, and solve for the unit step response of the resulting analog system.



The system after removing sampler and ZOH blockes

Therefore, the closed loop transfer function of the continuous system is;

$$T_a(s) = \frac{G_p(s)}{1 + G_p(s)} = \frac{4}{s + 6}$$

where $G_p(s) = 4/(s + 2)$ is the plant transfer function. Hence the analog system unit-step response is given by

$$C_a(s) = \frac{4}{s(s + 6)} = \frac{0.667}{s} + \frac{-0.667}{s + 6}$$

and

$$c_a(t) = 0.667(1 - \epsilon^{-6t})$$

The response is also shown at Fig.1.b.

Example 3.2:

Find the discrete time and continuous time response for the system shown in Fig. 3.2 for sampling time T=1 sec, and compare between the two.

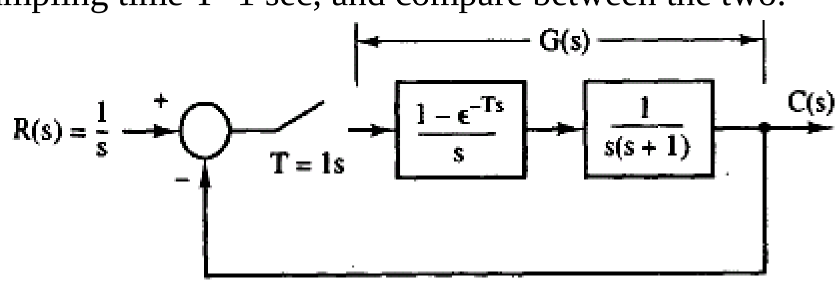


Fig. 3.2 System block diagram

By analysis;

$$C(z) = \frac{G(z)}{1 + G(z)}R(z)$$

Also it can be shown that;

$$G(z) = \left(\frac{z-1}{z}\right)\mathcal{Z}\left[\frac{1}{s^2(s+1)}\right]_{T=1} = \frac{z-1}{z}\left[\frac{z[(1-1+\epsilon^{-1})z + (1-\epsilon^{-1}-\epsilon^{-1})]}{(z-1)^2(z-\epsilon^{-1})}\right] = \frac{0.368z + 0.264}{z^2 - 1.368z + 0.368}$$

Then $$\frac{C(z)}{R(z)} = \frac{G(z)}{1+G(z)} = \frac{0.368z + 0.264}{z^2 - z + 0.632}$$

Since $$R(z) = \frac{z}{z-1}$$

then $$C(z) = \frac{z(0.368z + 0.264)}{(z-1)(z^2 - z + 0.632)} =$$

using long division ;

$$C(z) = 0.368z^{-1} + 1.00z^{-2} + 1.40z^{-3} + 1.40z^{-4} + 1.15z^{-5} + 0.90z^{-6} + 0.80z^{-7} + 0.87z^{-8}$$
$$+ 0.99z^{-9} + 1.08z^{-10} + 1.08z^{-11} + 1.00z^{-12} + 0.98z^{-13} + \cdots$$

The final value of $c(nT)$, obtained using the final-value theorem, is $\lim_{n\to\infty} c(nT) = \lim_{z\to 1}(z-1)C(z) = \frac{0.632}{0.632} = 1$

The step response for this system is plotted in Fig.3. The response between sampling instants was obtained from a simulation of the system. Moreover, the continuous response of the system (without sampler and ZOH) was also plotted in Fig. 3.3 for comparison.

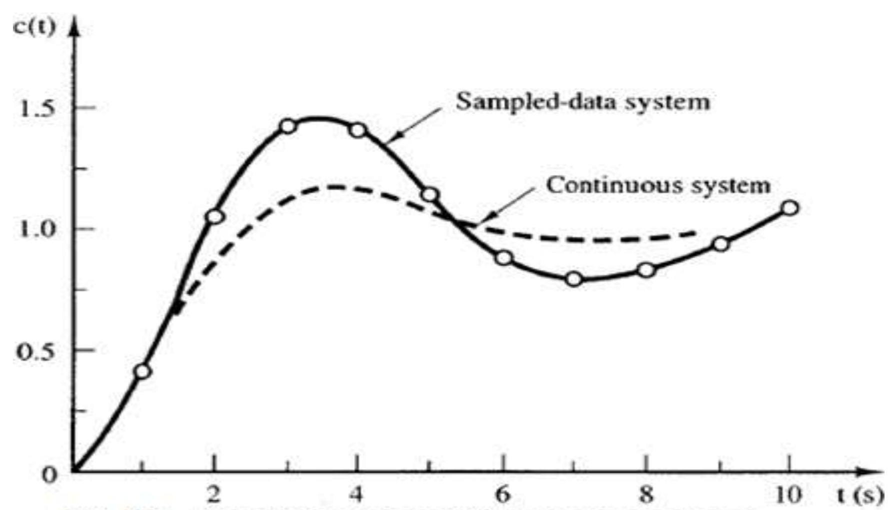


Fig. 3.3 Discrete and continuous responses

3.2 Review of Time Response

The Laplace transform of the unit impulse is $R(s) = 1$, and therefore the output for an impulse is

$$Y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2},$$

which is $T(s) = Y(s)/R(s)$, the transfer function of the closed-loop system. The transient response for an impulse function input is then

$$y(t) = \frac{\omega_n}{\beta} e^{-\zeta\omega_n t} \sin(\omega_n \beta t),$$

where $\beta = \sqrt{1 - \zeta^2}$, $\theta = \cos^{-1}\zeta$, and $0 < \zeta < 1$.

which is the derivative of the response to a step input. The impulse response of the second-order system is shown in Figure 3.4 for several values of the damping ratio ξ. The designer is able to select several alternative performance measures from the transient response of the system for either a step or impulse input.

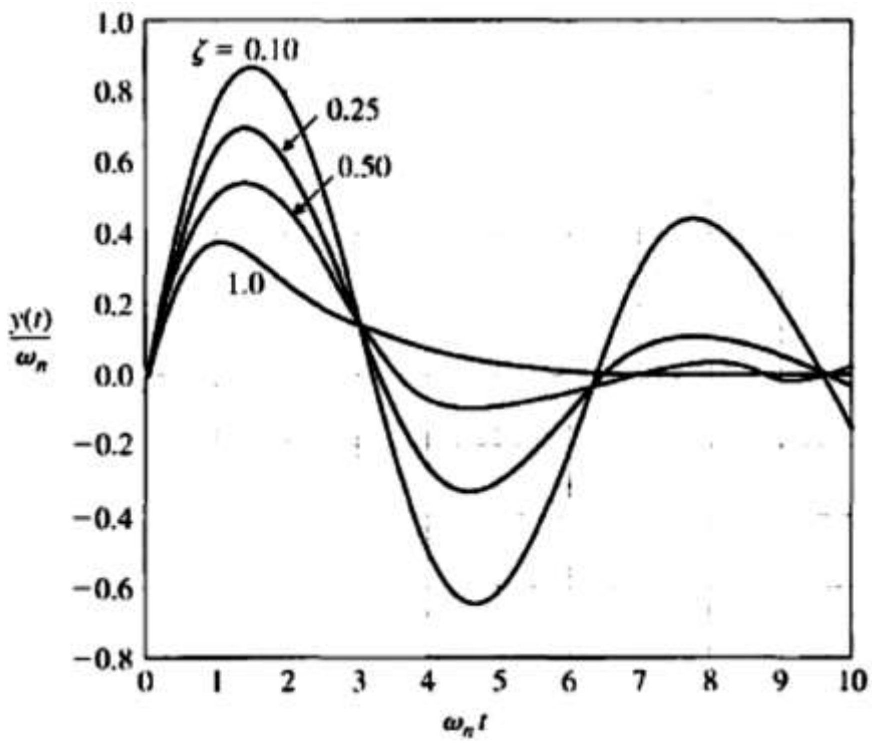


Figure 3.4: Response of a 2^{nd} order system to an impulse input.

Standard performance measures are usually defined in terms of the step response of a system as shown in Figure 3.5. The swiftness of the response is measured by the **rise time *Tr*** and the **peak time** *Tp.* For under-damped systems with an overshoot, the 0-100% rise time is a useful index. If the system is over-damped, then the peak time is not defined, and the 10-90% rise time ***Tr***, is normally used. The similarity with which the actual response matches the step input is measured by the percent overshoot and settling time *Ts.* The **percent overshoot** is defined as

$$\text{percentage overshoot} = \frac{\text{Maximum Peak} - \text{Final Value}}{\text{Final Value}}\ x\,100$$

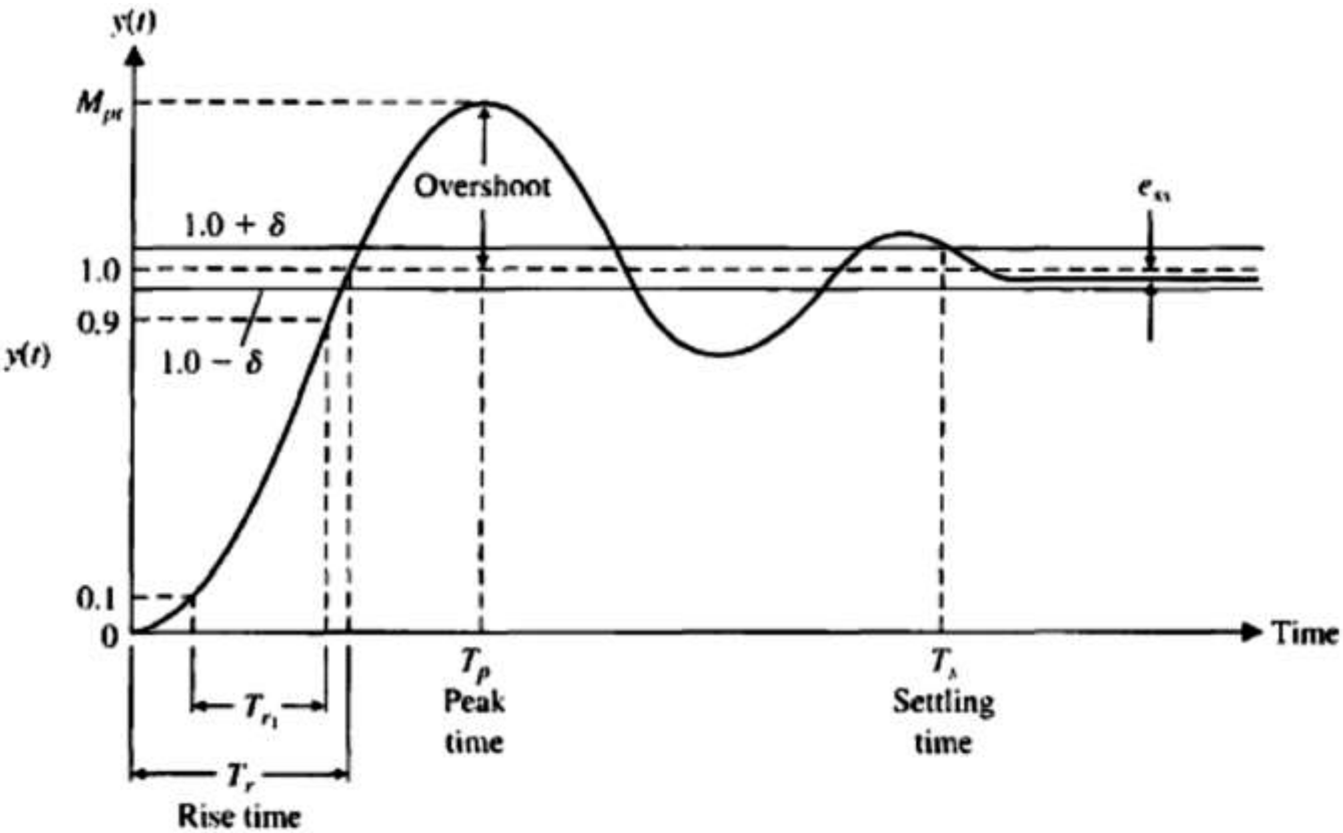


Figure 3.5: Step Response of a control system.

The **settling time,** *Ts,* is defined as the time required for the system to settle within a certain percentage δ of the input amplitude. This band of $\pm\,\delta$ is shown in Figure 5. For the second-order system with closed-loop damping constant $\zeta\omega_n$ and a response described by

$$y(t) = 1 - \frac{1}{\beta}e^{-\zeta\omega_n t}\sin(\omega_n\beta t + \theta)$$

we seek to determine the time *Ts* for which the response remains within **2%** of the final value. This occurs approximately when

$$\boxed{T_s = \frac{4}{\zeta\omega_n}.}$$

Also; the peak time is

$$\boxed{T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}},}$$

And the percentage over shoot is l

$$\boxed{P.O. = 100e^{-\zeta\pi/\sqrt{1-\zeta^2}}.}$$

Also it might be written in the opposite way;

$$\zeta = \sqrt{\frac{\ln^2(P.O)}{\pi^2 + \ln^2(P.O)}}$$

And poles are located at

$$S_d = \sigma \mp \omega_d = -\zeta\omega_n \mp j\,\omega_n\sqrt{1-\zeta^2}$$

Example 3.3: The open loop gain of a closed loop system with unity feedback is;

$$L(S) = G_C(S)G(S) = \frac{2(S+8)}{S(S+4)}$$

(a) Determine the closed loop transfer function T(S)=Y(S)/R(S).
(b) Calculate the settling, and peak time values.
(c) Calculate the percentage of maximum over-shoot.

Solution:
(a)

$$T(S) = \frac{G_C(S)G(S)}{1+G_C(S)G(S)} = \frac{\frac{2(S+8)}{S(S+4)}}{1+\frac{2(S+8)}{S(S+4)}} = \frac{2(S+8)}{S(S+4)+2(S+8)} = \frac{2(S+8)}{S^2+6S+8}$$

From the characteristic equation S^2+6S+8 it can be deduced that ω_n=2.828 rad/sec and then ζ=0.17677.
(b)

$$T_S = \frac{4}{0.17677 * 2.828} = 8\ sec$$

and

$$T_P = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 1.1286\ sec$$

(c)

$$P.O. = 100 * e^{-\zeta\pi / \sqrt{1-\zeta^2}}$$

Then P.O.=56.8%

3.3 Mapping the S-Plane into the Z-Plane

To introduce the topic, consider a function $e(t)$, which is sampled with the resulting starred transform $E^*(s)$. At the sampling instants, the sampled signal is of the same nature (and has the same values) as the continuous signal. For example, if $e(t)$ is exponential, then the sampled signal is exponential at the sampling instants, with the same amplitude and time constant as the continuous function. If $e(t) = \epsilon^{-at}$,

$$E(s) = \frac{1}{s+a}, \quad E(z) = \frac{z}{z - \epsilon^{-aT}}$$

$s = \sigma \pm j\,\omega$

Considering only the left hand side (Stable region);

$and\ since \quad z = e^{sT} \quad then \quad z = e^{(\sigma + j\,\omega)T} = e^{T\sigma} . e^{j\,\omega T}$

$using\ the\ identity \qquad e^{j\,\omega T} = \cos \omega T + j \sin \omega T$

$then;$ $$\boldsymbol{z = e^{T\sigma} \cos \omega T + j\, e^{T\sigma} \sin \omega T}$$

Also

$$\boldsymbol{z = e^{T\sigma} \angle \omega T}$$

Consider first the mapping of the left half-plane portion of the primary strip into the z-plane as shown in Figure 3.6.

$The\ translation\ between\ the\ s-and\ z-plane\ is\ considerd\ here\ for$ the switching frequency $\boldsymbol{\omega_s}$

$$\boldsymbol{\omega = \omega_s = 2\pi\, f_s = 2\pi \frac{1}{T}}$$

Therefore;

$$\omega T = \boldsymbol{2\pi}$$

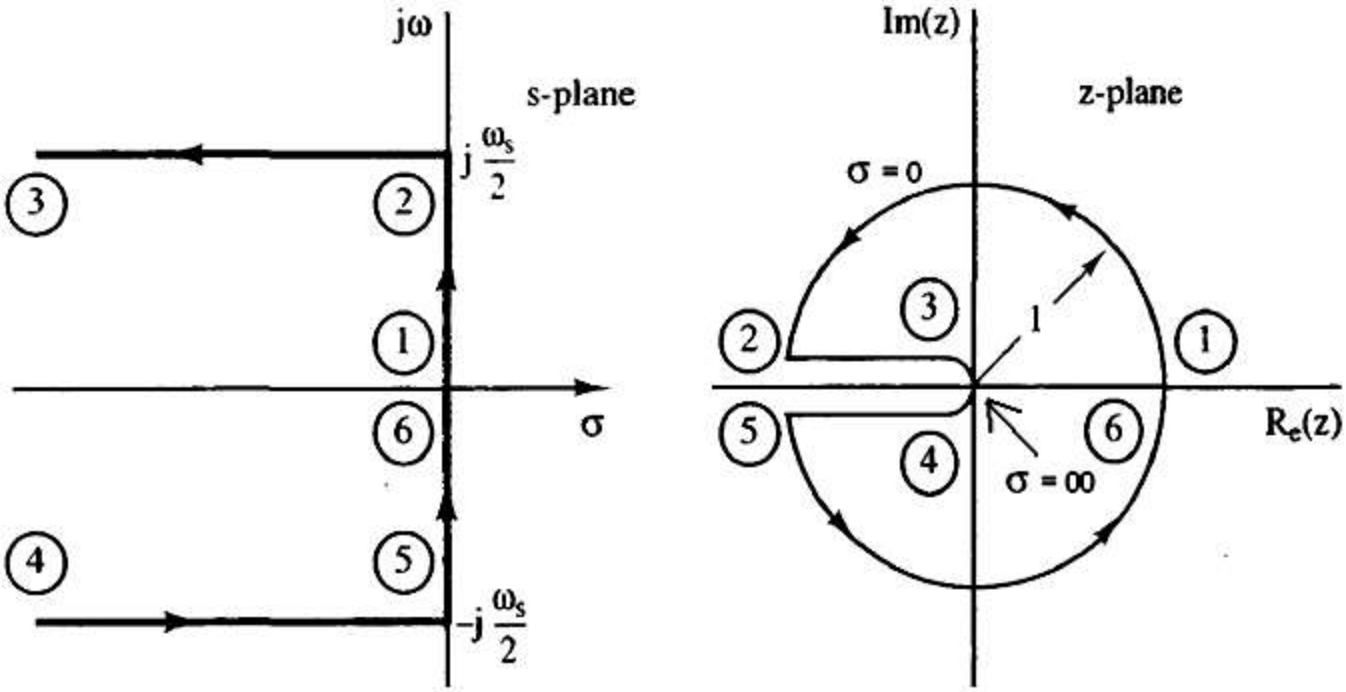


Figure 3.6 Mapping the primary strip into the z-plane.

Figure 3.7 shows some cases of the effect of changing the real and imaginary parts in the s-plane on the z-plane plot.

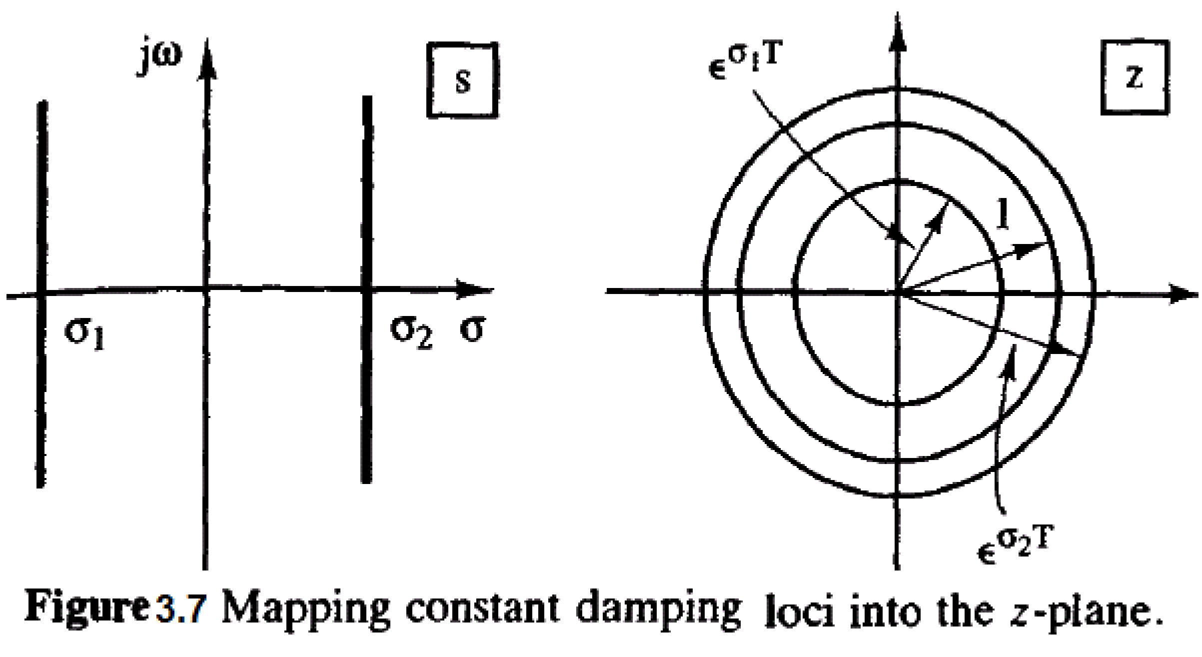


Figure 3.7 Mapping constant damping loci into the z-plane.

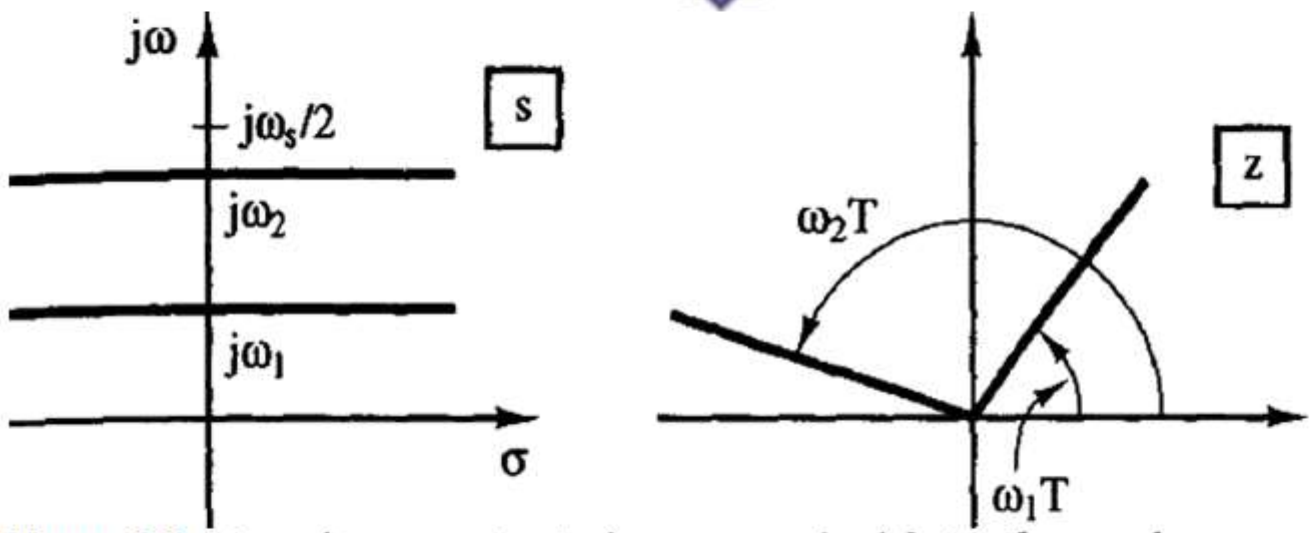


Figure 3.8 Mapping constant frequency loci into the z-plane.

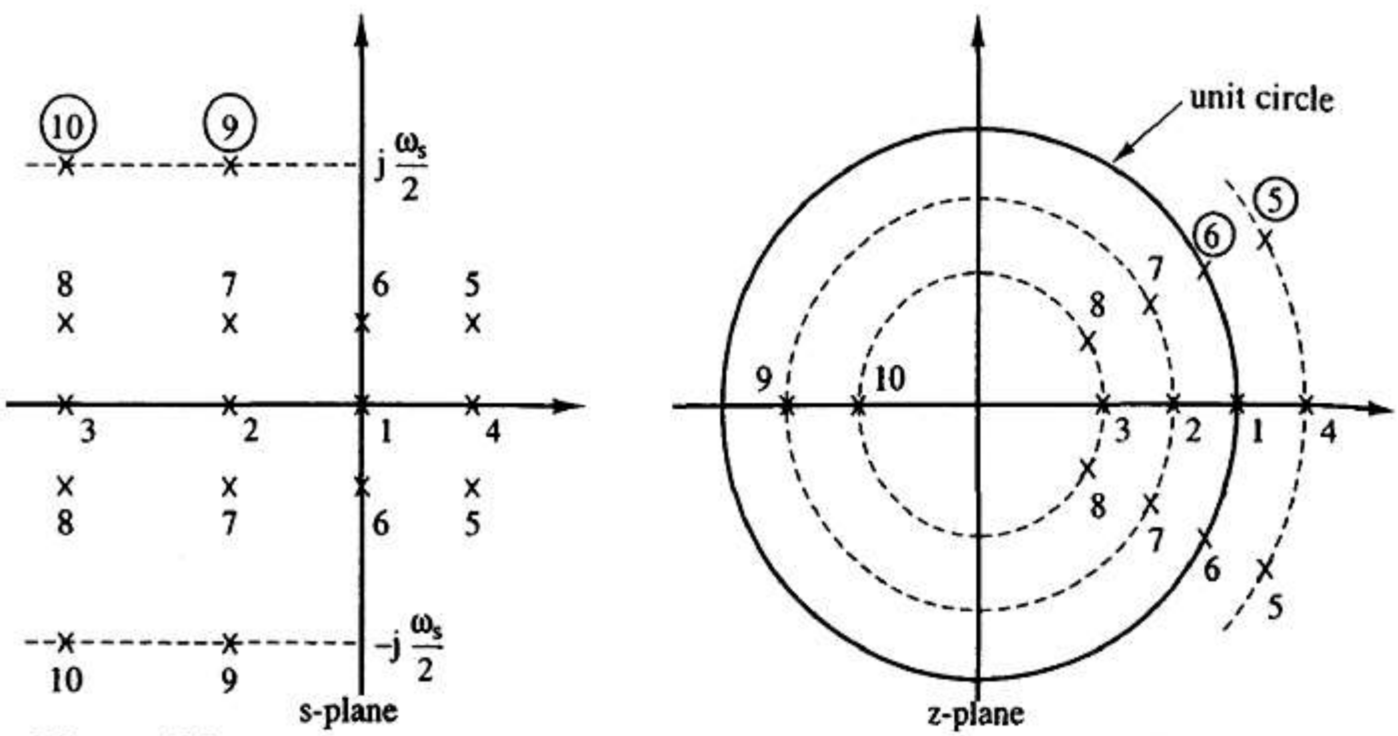


Figure 3.9 Corresponding pole locations between the s-plane and the z-plane.

Figure 3.9 was plotted where these s-plane poles result in z-plane poles at

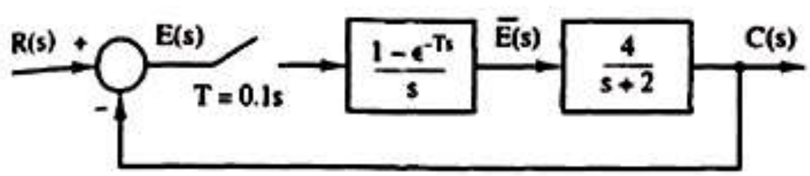
$$z = \epsilon^{sT}\Big|_{s=\sigma \pm j\omega} = \epsilon^{\sigma T}\epsilon^{\pm j\omega T} = \epsilon^{\sigma T}\underline{/\pm\omega T} = r\underline{/\pm\theta}$$

Thus roots of the characteristic equation that appear at $z = r\angle\pm\theta$ result in a transient-response term of the form

$$A\epsilon^{\sigma kT}\cos(\omega kT + \phi) = A(r)^k\cos(\theta k + \phi)$$

Example 3.4:

Find the s-plane poles for the system shown below



The closed loop transfer function is found as ;

$$\frac{G(z)}{1 + G(z)} = \frac{0.3625}{z - 0.4562}$$

Hence the closed-loop characteristic equation is

$$z - 0.4562 = 0$$

Therefore; to determine the first pole (and the only one);

$$z_1 = 0.4562 = \epsilon^{s_1 T} = \epsilon^{0.1 s_1}$$

Hence $s_1 = \ln(0.4562)/0.1$, or $s_1 = -7.848$.

-- End of example

In figure 3.10 we may see and compare the stable versus the unstable regions in z-plane (inside and outside the unity circle) depending on examples of time response signals.

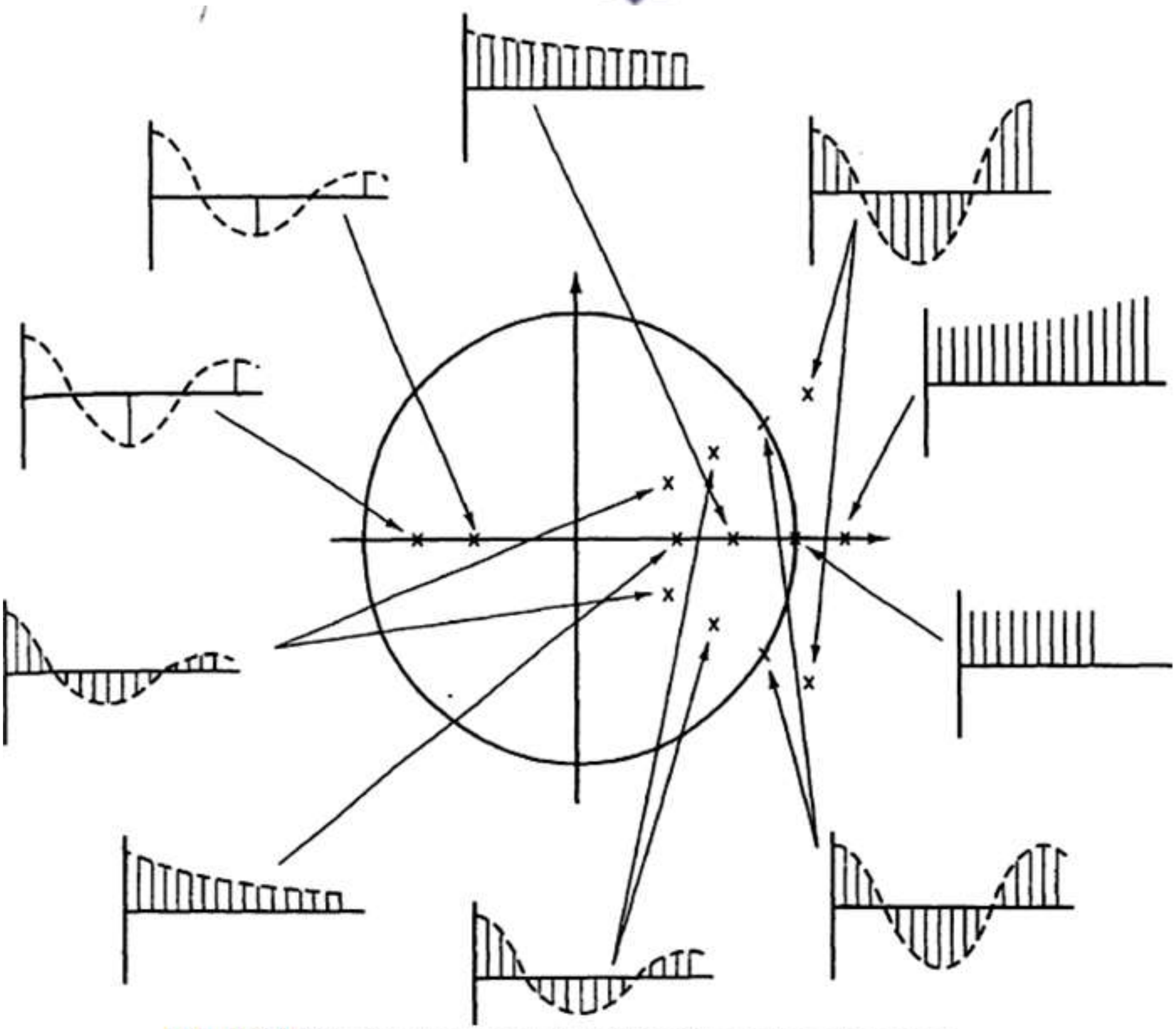


Figure 3.10 Transient response characteristics of the z-plane pole locations.

In the discussion above, we considered the relationship between s-plane poles and z-plane poles in a general way. We will now mathematically relate the s-plane pole locations and the z-plane pole locations. We express in standard form the s-plane second-order transfer function

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

which has the poles

$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

where ζ is the damping ratio and ω_n is the natural frequency. The equivalent z-plane poles occur at

$$z = \epsilon^{sT}\big|_{s_{1,2}} = \epsilon^{-\zeta\omega_n T}\angle \pm\omega_n T\sqrt{1-\zeta^2} = r\angle\pm\theta$$

Hence $\boxed{\epsilon^{-\zeta\omega_n T} = r}$

or $\zeta\omega_n T = -\ln r$ Also, $\boxed{\theta = \omega_n T\sqrt{1-\zeta^2}}$

Taking the ratio of the last two equations, we obtain $\dfrac{\zeta}{\sqrt{1-\zeta^2}} = \dfrac{-\ln r}{\theta}$

Solving this equation for ζ yields $\boxed{\zeta = \dfrac{-\ln r}{\sqrt{\ln^2 r + \theta^2}}}$

We then find ω_n to be $\boxed{\omega_n = \dfrac{1}{T}\sqrt{\ln^2 r + \theta^2}}$

The time constant, τ, of the poles is then given by $\boxed{\tau = \dfrac{1}{\zeta\omega_n} = \dfrac{-T}{\ln r}}$

This equation can also be expressed as $r = \epsilon^{-T/\tau}$

Thus, given the complex pole location in the z-plane, we find the damping ratio, the natural frequency, and the time constant of the pole ·

Example 3.5:

For the closed loop system transfer function below;

$$\frac{G(z)}{1+G(z)} = \frac{0.368z + 0.264}{z^2 - z + 0.632}, \qquad T = 1 \text{ s}$$

Thus the system characteristic equation is

$$z^2 - z + 0.632 = (z - 0.5 - j0.618)(z - 0.5 + j0.618) = 0$$

The poles are then complex and occur at

$$z = 0.5 \pm j0.618 = 0.795\underline{/\pm 51.0^\circ} = 0.795\underline{/\pm 0.890 \text{ rad}}$$

Therefore;

$$z = \epsilon^{\sigma T}\underline{/\pm\omega T} = r\underline{/\pm\omega T} = 0.795\underline{/\pm 0.890}$$

then

$$\zeta = \frac{-\ln(0.795)}{[\ln^2(0.795) + (0.890)^2]^{1/2}} = 0.250$$

$$\omega_n = \frac{1}{1}[\ln^2(0.795) + (0.890)^2]^{1/2} = 0.9191$$

$$\tau = \frac{-1}{\ln(0.795)} = 4.36 \text{ s}$$

-- End of example

3.4 Steady - State Accuracy

An important characteristic of a control system is its ability to follow, or track, certain inputs with a minimum of error. The control system designer attempts to minimize the system error to certain anticipated inputs. In this section the effects of the system transfer characteristics on the steady-state system errors are considered.

Consider the system shown in Figure 3.11.

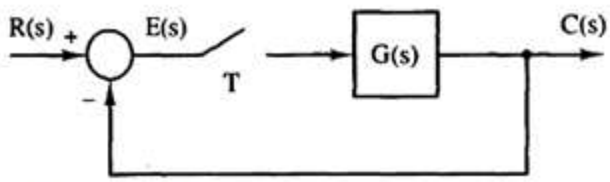


Figure 3.11 Discrete-time system.

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1 + G(z)}$$

where $G(z) = \mathfrak{z}[G(s)]$. The plant transfer function can always be expressed as

$$G(z) = \frac{K\prod^{m}(z - z_i)}{(z-1)^N\prod^{p}(z - z_j)}, \qquad z_i \neq 1, \qquad z_j \neq 1$$

As we shall see, the value of N has special significance and is called the *system type*. For convenience in the following development, we define

$$K_{dc} = \left.\frac{K\prod^{m}(z - z_i)}{\prod^{p}(z - z_j)}\right|_{z=1}$$

Note that K_{dc} is the open-loop plant dc gain with all poles at $z = 1$ removed.

the system error, $e(t)$, is defined as the difference between the system input and the system output. Or

$$E(z) = \mathfrak{z}[e(t)] = R(z) - C(z)$$

Then, by substituting this equation in eq. *, then $E(z) = R(z) - \frac{G(z)}{1 + G(z)}R(z) = \frac{R(z)}{1 + G(z)}$

The steady-state errors will now be derived for two common inputs—a position (step) input and a velocity (ramp) input. First, for the unit-step input,

$$R(z) = \frac{z}{z-1}$$

Then, from the final-value theorem, the steady-state error is seen to be

$$e_{ss}(kT) = \lim_{z \to 1} (z-1)E(z) = \lim_{z \to 1} \frac{(z-1)R(z)}{1+G(z)}$$

And since $R(z) = \frac{z}{z-1}$

provided that $e_{ss}(kT)$ has a final value. The steady-state error is then

$$e_{ss}(kT) = \lim_{z \to 1} \frac{z}{1+G(z)} = \frac{1}{1+\lim_{z \to 1} G(z)}$$

We now define the position error constant as

$$K_p = \lim_{z \to 1} G(z)$$

Then in , if $N = 0$ [i.e., no poles in $G(z)$ at $z = 1$], $K_p = K_{dc}$ and

$$e_{ss}(kT) = \frac{1}{1+K_p} = \frac{1}{1+K_{dc}}$$

For $N \geq 1$ (system type greater than or equal to one), $K_p = \infty$ and the steady-state error is zero.

Consider next the unit-ramp input. In this case $r(t) = t$, $R(z) = \frac{Tz}{(z-1)^2}$

and since $e_{ss}(kT) = \dfrac{1}{1+\lim_{z \to 1} G(z)}$

Therefore,

$$e_{ss}(kT) = \lim_{z\to 1} \frac{Tz}{(z-1)+(z-1)G(z)} = \frac{T}{\lim_{z\to 1}(z-1)G(z)}$$

We now define the velocity error constant as

$$K_v = \lim_{z\to 1} \frac{1}{T}(z-1)G(z) \quad \text{------} \quad *1$$

Then if $N = 0$, $K_v = 0$ and $e_{ss}(kT) = \infty$. For $N = 1$, $K_v = K_{dc}/T$ and

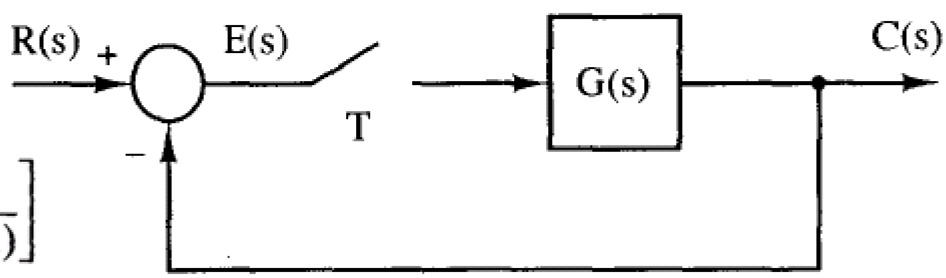
$$e_{ss}(kT) = \frac{1}{K_v} = \frac{T}{K_{dc}} \quad \text{------ --} \quad *2$$

For $N \geq 2$ (system type greater than or equal to 2), $K_v = \infty$ and $e_{ss}(kT)$ is zero.

The development above illustrates that, in general, increased system gain and/or the addition of poles at $z = 1$ to the open-loop forward-path transfer function tend to decrease steady-state errors.

<u>**Example 3.6:**</u>

Find the steady state error for the system shown below:



$$G(s) = \frac{1-\epsilon^{-Ts}}{s}\left[\frac{K}{s(s+1)}\right]$$

Thus

$$G(z) = K\mathfrak{z}\left[\frac{1-\epsilon^{-Ts}}{s^2(s+1)}\right] = \frac{K(z-1)}{z}\mathfrak{z}\left[\frac{1}{s^2(s+1)}\right]$$

$$= \frac{K(z-1)}{z}\,\frac{z[(\epsilon^{-T}+T-1)z+(1-\epsilon^{-T}-T\epsilon^{-T})]}{(z-1)^2(z-\epsilon^{-T})}$$

$$G(z) = \frac{K[(\epsilon^{-T} + T - 1)z + (1 - \epsilon^{-T} - T\epsilon^{-T})]}{(z - 1)(z - \epsilon^{-T})}$$

Then, from eq. *1 , the system is type 1 and

$$K_v = \frac{K_{dc}}{T} = \frac{K[(\epsilon^{-T} + T - 1) + (1 - \epsilon^{-T} - T\epsilon^{-T})]}{T(1 - \epsilon^{-T})} = K$$

Since $G(z)$ has one pole at $z = 1$, the steady-state error to a step input is zero, and to a ramp input is, from eq. *2 $e_{ss}(kT) = \frac{1}{K_v} = \frac{1}{K}$ provided that the system is stable.

-- End of example

Example 3.7:

As a second example, consider again the system shown in last example , where, for this example,

$$G(z) = \mathfrak{z}\left[\frac{1 - \epsilon^{-Ts}}{s(s + 1)}\right] = \frac{1 - \epsilon^{-T}}{z - \epsilon^{-T}}$$

Suppose that the design specification for this system requires that the steady-state error to a unit ramp input be less than 0.01. Thus, from eq *1 , it is necessary that the open-loop system be type 1 or greater and thus the open-loop function must have at least one pole at $z = 1$. Since the plant does not contain a pole at $z = 1$, a digital compensator of the form

$$D(z) = \frac{K_I z}{z - 1} + K_P$$

will be added, to produce the resultant system shown inFigure 3.12. The compensator, called a PI or proportional-plus-integral compensator, is of a form commonly used to reduce steady-state errors. For this system eq *1 becomes :

$$K_v = \lim_{z \to 1} \frac{1}{T}(z - 1)D(z)G(z)$$

Employing the expressions above for $D(z)$ and $G(z)$, we see that

$$K_v = \lim_{z \to 1} (z - 1) \frac{(K_I + K_P)z - K_P}{T(z - 1)} \left[\frac{1 - \epsilon^{-T}}{z - \epsilon^{-T}} \right] = \frac{K_I}{T}$$

Thus K_I equals $100T$ for the required steady-state error, provided that the system is stable. The latter point is indeed an important consideration since the error analysis is meaningless unless stability of the system is guaranteed.

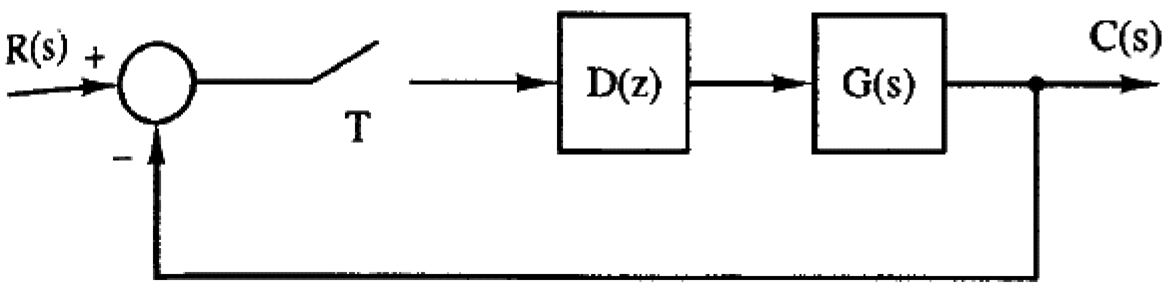


Figure 3.12 System with compansator