Stability Analysis Techniques

In this section the stability analysis techniques for the Linear Time-Invarient (LTI) discrete system are emphasized. In general the stability techniques applicable to LTI continuous-time systems may also be applied to the analysis of LTI discrete-time systems (if certain modifications are made).

4.1 Stability

To introduce the stability concept, consider the LTI system shown in Fig. 4.1. For this system;

$$C(z) = \frac{G(z)R(z)}{1 + \overline{GH}(z)} = \frac{K\prod^{m}(z - z_i)}{\prod^{n}(z - p_i)}R(z)$$

where z_i are the zeros and p_i the poles of the system transfer function. Using the partial-fraction expansion and for the case of distinct poles, we may write C(z) as;

$$C(z) = \frac{k_1 z}{z - p_1} + \cdots + \frac{k_n z}{z - p_n} + C_R(z)$$

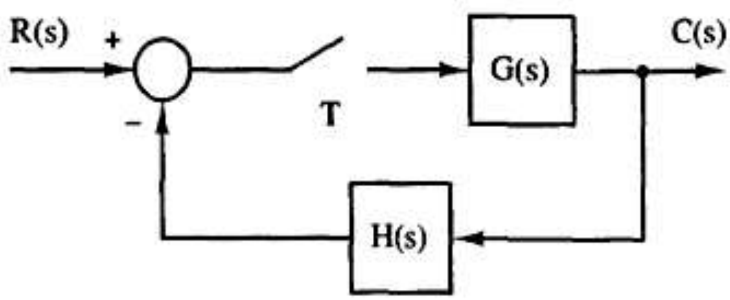


Fig.4.1 Sampled-data system

Where $C_R(z)$ contains the terms of C(z) which originate in the poles of R(z). The first n terms are the natural-response terms of C(z). If the inverse z-transform of these terms tend to zero as time increases, the system is considered as stable and these terms are called transient response. The z-transform of the i-th term is

$$\mathfrak{z}^{-1}\left[\frac{k_i z}{z - p_i}\right] = k_i(p_i)^k$$

Thus, if the magnitude of p_i is less than 1, this term approaches zero as k approaches infinity. Note that the factor $(z - p_i)$ originate in the characteristic equation of the system, that is, in

$$1 + \overline{GH}(z) = 0$$

The system is stable provided that all the roots lie inside the unit circle in the z-plane.

4.2 Bilinear Transformation

Many analysis and design techniques for continuous time LTI systems, such as the Routh-Hurwitz criterion and Bode technique, are based on the property that in the s-plane the stability boundary is the imaginary axis. These techniques cannot be applied to LTI discrete-time system in the z-plane, since the stability boundary is the unit circle. However, through the use of the following transformation;

$$z = \frac{1+\left(\frac{T}{2}\right)L}{1-\left(\frac{T}{2}\right)L}$$ (Transforming from z-plane to L-plane)

Or

$$L = \frac{2}{T} * \frac{z-1}{z+1}$$ (Transforming from L-plane to z-plane)

The unit circle of the z-plane transforms into the imaginary axis of L-plane. This can be seen through the following development. On the unit circle in the z-plane,

$$z = e^{j\omega T}$$

and then substitute in the transformation formula;

$$L = \frac{2}{T}\frac{z-1}{z+1}\bigg|_{z=\epsilon^{j\omega T}} = \frac{2}{T}\frac{\epsilon^{j\omega T}-1}{\epsilon^{j\omega T}+1} = \frac{2}{T}\frac{\epsilon^{j\omega T/2}-\epsilon^{-j\omega T/2}}{\epsilon^{j\omega T/2}+\epsilon^{-j\omega T/2}}$$

which will result in;

$$\boxed{L = j\frac{2}{T}\tan\frac{\omega T}{2}}$$

Thus it is seen that the unit circle of the z-plane transformation into the imaginary axis of the L-plane. The mapping of the primary strip of the s-plane into both the z-plane ($z=e^{sT}$) and the L-plane are shown in Figure 4.2. It is noted that the stable region of the L-plane is the left half-plane.

Let the $j\omega_L$ be the imaginary part of L. We will refer to ω_L as the L-plane frequency. Then;

$$\boxed{\omega_L = \frac{2}{T}\tan\frac{\omega T}{2}}$$

and this expression gives the relationship between frequencies in the s-plane and frequencies in the L-plane. Now, for small values od real frequency (s-plane frequency) such that ωT is small,

$$\omega_L = \frac{2}{T}\frac{\omega T}{2} = \omega$$

Thus the L-plane frequency is approximately equal to the s-plane frequency for this case. The approximation is valid for those values of frequency for which $\tan(\omega T/2) \cong \omega T/2$. Now for;

$$\frac{\omega T}{2} \leq \frac{\pi}{10} \quad ,then\ \omega \leq \frac{2\pi}{10T} = \frac{\omega_s}{10}$$

The error in this appromimation is less than 4 percentage.

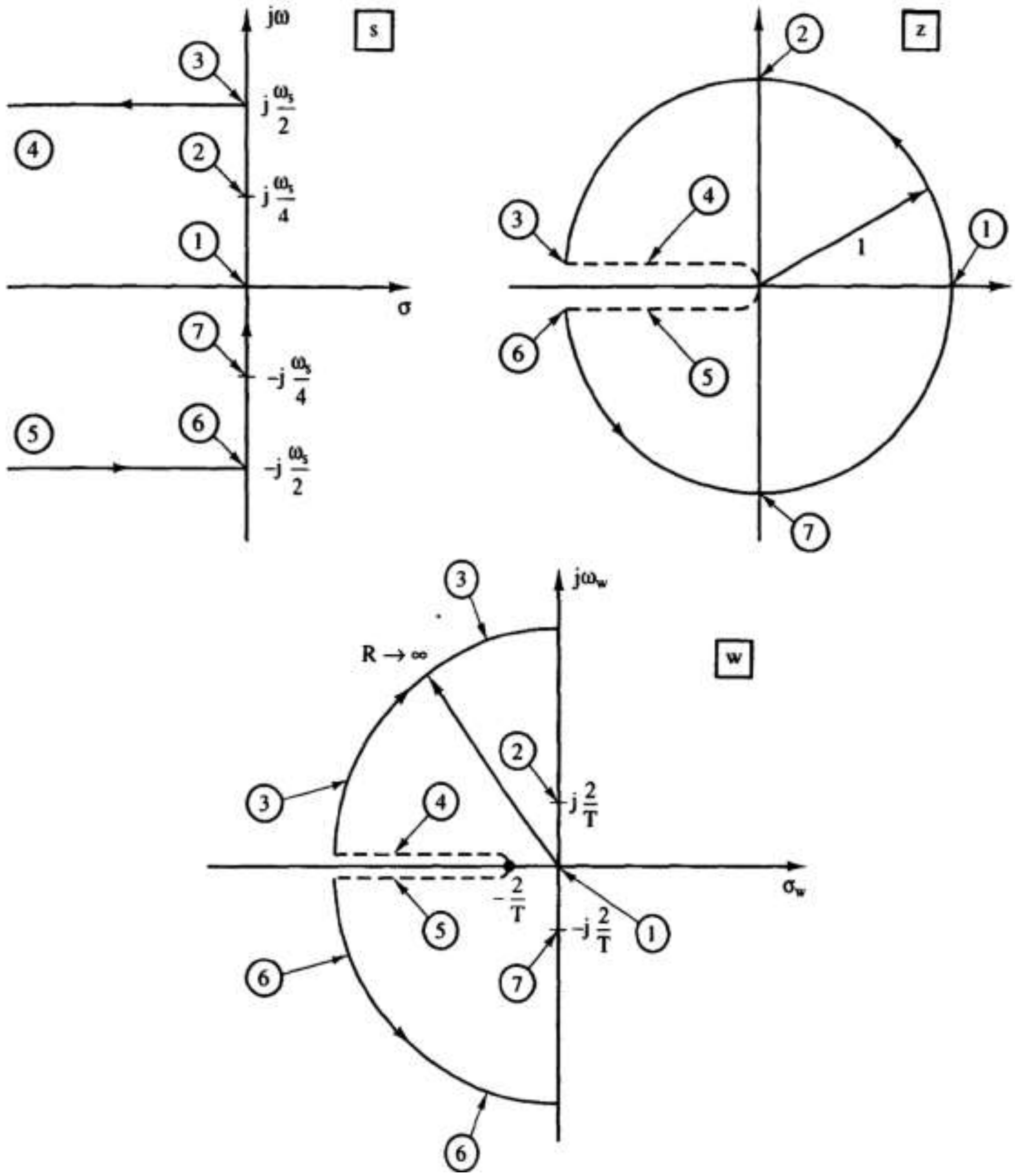


Figure 4.2 Mapping from s-plane to z-plane to L-plane

4.3 The Routh – Hurwitz Criterion

The Routh-Hurwitz criterion may be used in the analysis of LTI continuous-time system to determine if any roots of a given equation are in the RIGHT half side of the s-plane. If this criterion applied to the characteristic equation of an LTI discrete time system when expressed as a function of z, no useful information on stability is obtained. However, if the characteristic equation is expressed as a function of the bilinear transform variable (L), then the stability of the system may be determined using directly applying the Routh-Hurwitz criterion. The procedure of the criterion is shown briefly in Table 4.1.

Example 4.1:

Consider the system shown in Figure 4.3, check the system stability using Routh-Herwits criterion.

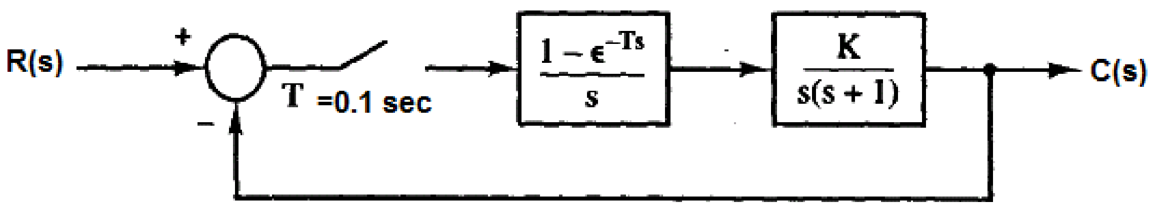


Figure 4.3

The Open-Loop function is;

$$G(s) = \frac{1 - e^{-Ts}}{s}\left[\frac{1}{s(s+1)}\right]$$

After obtaining the z-transform,

$$G(z) = \frac{z-1}{z}\left[\frac{(e^{-T} + T - 1)z^2 + (1 - e^{-T} - Te^{-T})z}{(z-1)^2\,(z - e^{-T})}\right]$$

$$G(z) = \frac{0.0048z + 0.00468}{(z-1)(z-0.905)}$$

Using Bilinear transformation;

$G(L) = G(z)$ $\quad$ at $Z = \frac{1+\left(\frac{T}{2}\right)L}{1-\left(\frac{T}{2}\right)L} = \frac{1+0.05L}{1-0.05L}$ $\quad$ Therefore;

$$G(L) = \frac{-0.00016\,L^2 - 0.1872\,L + 3.81}{3.81\,L^2 + 3.8\,L}$$

TABLE 4.1 BASIC PROCEDURE FOR APPLYING THE ROUTH–HURWITZ CRITERION

1. Given a characteristic equation of the form

$$F(w) = b_n \llcorner^n + b_{n-1} \llcorner^{n-1} + \cdots + b_1 \llcorner + b_0 = 0$$

form the Routh array as

$$\begin{array}{c|cccc}
\llcorner^n & b_n & b_{n-2} & b_{n-4} & \cdots \\
\llcorner^{n-1} & b_{n-1} & b_{n-3} & b_{n-5} & \cdots \\
\llcorner^{n-2} & c_1 & c_2 & c_3 & \cdots \\
\vdots & d_1 & d_2 & d_3 & \cdots \\
\llcorner^1 & j_1 & & & \\
\llcorner^0 & k_1 & & &
\end{array}$$

2. Only the first two rows of the array are obtained from the characteristic equation. The remaining rows are calculated as follows.

$$c_1 = \frac{b_{n-1} b_{n-2} - b_n b_{n-3}}{b_{n-1}} \qquad d_1 = \frac{c_1 b_{n-3} - b_{n-1} c_2}{c_1}$$

$$c_2 = \frac{b_{n-1} b_{n-4} - b_n b_{n-5}}{b_{n-1}} \qquad d_2 = \frac{c_1 b_{n-5} - b_{n-1} c_3}{c_1}$$

$$c_3 = \frac{b_{n-1} b_{n-6} - b_n b_{n-7}}{b_{n-1}} \qquad \vdots$$

3. Once the array has been formed, the Routh–Hurwitz criterion states that the number of roots of the characteristic equation with positive real parts is equal to the number of sign changes of the coefficients in the first column of the array.
4. Suppose that the w^{i-1}th row contains only zeros, and that the w^ith row directly above it has the coefficients $\alpha_1, \alpha_2, \ldots$. The auxiliary equation is then

$$\alpha_1 \llcorner^i + \alpha_2 \llcorner^{i-2} + \alpha_3 \llcorner^{i-4} + \cdots = 0$$

This equation is a factor of the characteristic equation.

Then the characteristic equation is given by

$$1 + KG(L) = (3.81 - 0.00016K)L^2 + (3.80 - 0.1872K)L + 3.81K = 0$$

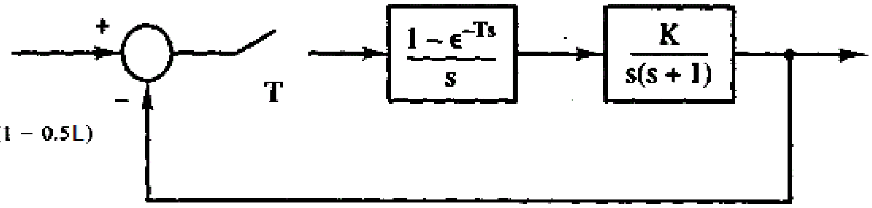
The Routh array derived from this equation is

$$\begin{array}{c|ll l}
L^2 & 3.81 - 0.00016K & 3.81K & \Rightarrow \quad K < 23{,}813 \\
L^1 & 3.80 - 0.1872K & & \Rightarrow \quad K < 20.3 \\
L^0 & 3.81K & & \Rightarrow \quad K > 0
\end{array}$$

Hence, for no sign changes to occur in the first column, it is necessary that K be in the range $0 < K < 20.3$, and this is the range of K for stability.

Examle 4.2

Consider the system shown with $T = 1\,s$ with a gain factor K added to the plant. The characteristic equation is given by



$$1 + KG(L) = 1 + KG(z)\big|_{z=(1+0.5L)/(1-0.5L)}$$

$$= 1 + \frac{K\left[0.368\left[\dfrac{1+0.5L}{1-0.5L}\right] + 0.264\right]}{\left[\dfrac{1+0.5L}{1-0.5L}\right]^2 - 1.368\left[\dfrac{1+0.5L}{1-0.5L}\right] + 0.368}$$

or $$1 + KG(L) = 1 + \frac{-0.0381K(L-2)(L+12.14)}{L(L+0.924)}$$

$$= \frac{(1 - 0.0381K)L^2 + (0.924 - 0.386K)L + 0.924K}{L(L+0.924)}$$

Thus the characteristic equation may be expressed as

$$(1 - 0.0381K)\,L^2 + (0.924 - 0.386K)\,L + 0.924K = 0$$

The Routh array is then

$$\begin{array}{c|lll} L^2 & 1 - 0.0381K & 0.924K & \Rightarrow \quad K < 26.2 \\ L^1 & 0.924 - 0.386K & & \Rightarrow \quad K < 2.39 \\ L^0 & 0.924K & & \Rightarrow \quad K > 0 \end{array}$$

Hence the system is stable for $0 < K < 2.39$.

From our knowledge of continuous-time systems, we know that the Routh–Hurwitz criterion can be used to determine the value of K at which the root locus crosses into the right half-plane (i.e., the value of K at which the system becomes unstable). That value of K is the gain at which the system is *marginally stable*, and thus can also be used to determine the resultant frequency of steady-state oscillation. Therefore, $K = 2.39$ in above Example is the gain for which the system is marginally stable.

In a manner similar to that employed in continuous-time systems, the frequency of oscillation at $K = 2.39$ can be found from the L^2 row of the array. Recalling that ω_w is the imaginary part of L, we obtain the auxiliary equation (see Table 4.1)

$$(1 - 0.0381K)\,L^2 + 0.924K\big|_{K=2.39} = 0.9089\,L^2 + 2.181 = 0$$

or

$$L = \pm j\sqrt{\frac{2.181}{0.9089}} = \pm j1.549$$

Then since $\omega_L = 1.549$

$$\omega = \frac{2}{T}\tan^{-1}\frac{\omega_w T}{2} = \frac{2}{1}\tan^{-1}\left[\frac{(1.549)(1)}{2}\right] = 1.32 \text{ rad/s}$$

and is the s-plane (real) frequency at which this system will oscillate with $K = 2.39$.

4.4 Jury's Stability Test

For continuous-time systems, the Routh–Hurwitz criterion offers a simple and convenient technique for determining the stability of low-ordered systems. However, since the stability boundary in the z-plane is different from that in the s-plane, the Routh–Hurwitz criterion cannot be directly applied to discrete-time systems if the system characteristic equation is expressed as a function of z. A stability criterion for discrete-time systems that is similar to the Routh–Hurwitz criterion and can be applied to the characteristic equation written as a function of z is the Jury stability test .

Jury's test will now be presented. Let the characteristic equation of a discrete-time system be expressed as

$$Q(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0, \qquad a_n > 0$$

Then form the array as shown in Table 2. Note that the elements of each of the

TABLE 4.2 ARRAY FOR JURY'S STABILITY TEST

z^0	z^1	z^2	$\cdots$	z^{n-k}	$\cdots$	z^{n-1}	z^n
a_0	a_1	a_2	$\cdots$	a_{n-k}	$\cdots$	a_{n-1}	a_n
a_n	a_{n-1}	a_{n-2}	$\cdots$	a_k	$\cdots$	a_1	a_0
b_0	b_1	b_2	$\cdots$	b_{n-k}	$\cdots$	b_{n-1}	
b_{n-1}	b_{n-2}	b_{n-3}	$\cdots$	b_{k-1}	$\cdots$	b_0	
c_0	c_1	c_2	$\cdots$	c_{n-k}	$\cdots$		
c_{n-2}	c_{n-3}	c_{n-4}	$\cdots$	c_{k-2}	$\cdots$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$			
l_0	l_1	l_2	l_3				
l_3	l_2	l_1	l_0				
m_0	m_1	m_2					

even-numbered rows are the elements of the preceding row in reverse order. The elements of the odd-numbered rows are defined as

$$b_k = \begin{vmatrix} a_0 & a_{n-k} \\ a_n & a_k \end{vmatrix}, \qquad c_k = \begin{vmatrix} b_0 & b_{n-1-k} \\ b_{n-1} & b_k \end{vmatrix}$$

$$d_k = \begin{vmatrix} c_0 & c_{n-2-k} \\ c_{n-2} & c_k \end{vmatrix} \ldots$$

The necessary and sufficient conditions for the polynomial $Q(z)$ to have no roots outside or on the unit circle, with $a_n > 0$, are as follows:

$$Q(1) > 0$$

$$(-1)^n Q(-1) > 0$$

$$|a_0| < a_n$$

$$|b_0| > |b_{n-1}|$$

$$|c_0| > |c_{n-2}| \qquad \text{------------------- } *$$

$$|d_0| > |d_{n-3}|$$

$$\vdots$$

$$|m_0| > |m_2|$$

Note that for a second-order system, the array contains only one row. For each additional order, two additional rows are added to the array. Note also that for an nth-order system, there are a total of $n + 1$ constraints.

Jury's test may be applied in the following manner:

1. Check the three conditions $Q(1) > 0$, $(-1)^n Q(-1) > 0$, and $|a_0| < a_n$, which requires no calculations. Stop if any of these conditions are not satisfied.
2. Construct the array, checking the conditions of (eq.*) as each row is calculated. Stop if any condition is not satisfied.

Example 4.3

Consider again the system below . Suppose that a gain factor K is added to the plant, and it is desired to determine the range of K for which the system is stable. Now, the system characteristic equation is

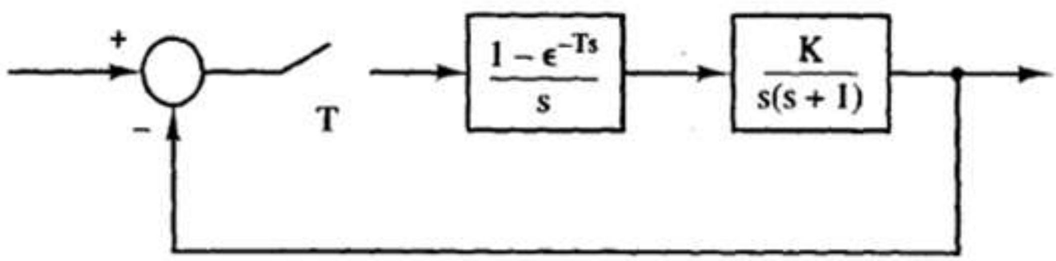
$$1 + KG(z) = 1 + \frac{(0.368z + 0.264)K}{z^2 - 1.368z + 0.368} = 0$$

or

$$z^2 + (0.368K - 1.368)z + (0.368 + 0.264K) = 0$$

The Jury array is

z^0	z^1	z^2
$0.368 + 0.264K$	$0.368K - 1.368$	1



The constraint $Q(1) > 0$ yields

$$1 + (0.368K - 1.368) + (0.368 + 0.264K) = 0.632K > 0 \Rightarrow K > 0$$

The constraint $(-1)^2 Q(-1) > 0$ yields

$$1 - 0.368K + 1.368 + 0.368 + 0.264K > 0 \Rightarrow K < \frac{2.736}{0.104} = 26.3$$

The constraint $|a_0| < a_2$ yields

$$0.368 + 0.264K < 1 \Rightarrow K < \frac{0.632}{0.264} = 2.39$$

Thus the system is stable for

$$\boxed{0 < K < 2.39}$$

The system is marginally stable for $K = 2.39$. For this value of K, the characteristic equation is

$$z^2 + (0.368K - 1.368)z + (0.368 + 0.264K)|_{K=2.39} = z^2 - 0.488z + 1 = 0$$

The roots of this equation are

$$z = 0.244 \pm j0.970 = 1\angle \pm 75.9° = 1\angle \pm 1.32 \text{ rad} = 1\angle \pm \omega T$$

Since $T = 1$ s, the system will oscillate at a frequency of 1.32 rad/s.

Example 4.4

Suppose that the characteristic equation for a closed-loop discrete-time system is given by the expression

$$\boxed{Q(z) = z^3 - 1.8z^2 + 1.05z - 0.20 = 0}$$

The first conditions of Jury's test are

$$Q(1) = 1 - 1.8 + 1.05 - 0.2 = 0.05 > 0$$

$$(-1)^3 Q(-1) = -[-1 - 1.8 - 1.05 - 0.2] = 4.05 > 0$$

$$|a_0| = 0.2 < a_3 = 1$$

The Jury array is calculated to be

z^0	z^1	z^2	z^3
−0.2	1.05	−1.8	1
1	−1.8	1.05	−0.2
−0.96	1.59	−0.69	

where the last row has been calculated as follows:

$$b_0 = \begin{vmatrix} -0.2 & 1 \\ 1 & -0.2 \end{vmatrix} = -0.96, \qquad b_1 = \begin{vmatrix} -0.2 & -1.8 \\ 1 & 1.05 \end{vmatrix} = 1.59$$

Hence the last condition is

$$|b_0| = 0.96 > |b_2| = 0.69$$

Therefore, since all conditions are satisfied, the system is stable. The characteristic equation can be factored as

$$Q(z) = (z - 0.5)^2(z - 0.8)$$

This form of the equation clearly indicates the system's stability.

Example 4.5

In this example we wish to determine the range of values of K_p . The characteristic equation for the system is given by the expression

$$1 + \frac{(K_I + K_p)z - K_p}{z - 1}\left[\frac{1 - \epsilon^{-T}}{z - \epsilon^{-T}}\right] = 0$$

Thus

$$z^2 - [(1 + \epsilon^{-T}) - (1 - \epsilon^{-T})(K_I + K_p)]z + \epsilon^{-T} - (1 - \epsilon^{-T})K_p = 0$$

Then the characteristic equation becomes

$$z^2 - (0.953 - 0.0952K_p)z + 0.905 - 0.0952K_p = 0$$

For this system, the Jury array is

z^0	z^1	z^2
$0.905 - 0.0952K_p$	$0.0952K_p - 0.953$	1

The constraint $Q(1) > 0$ yields

$$1 + 0.0952K_p - 0.953 + 0.905 - 0.0952K_p > 0$$

Thus this constraint is satisfied independent of K_p. The constraint $(-1)^2 Q(-1) > 0$ yields

$$1 - 0.0952K_p + 0.953 + 0.905 - 0.0952K_p > 0$$

or since K_p is normally positive,

$$0 < K_p < 15.01$$

The constraint $|a_0| < a_2$ yields

$$|0.905 - 0.0952K_p| < 1$$

or

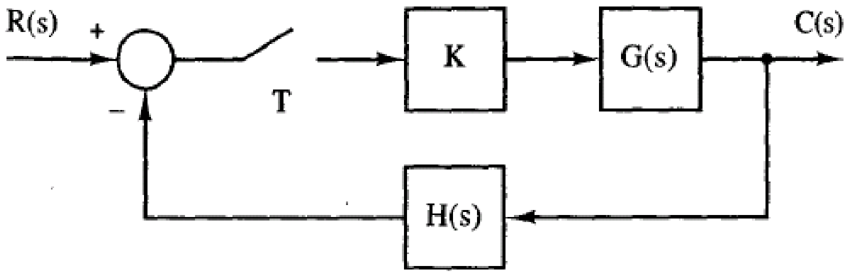
$$K_p < 20.0$$

Thus the stability constraint on positive K_p is $K_p < 15.01$

4.5 ROOT LOCUS

For the LTI sampled-data system of Figure below

$$\frac{C(z)}{R(z)} = \frac{KG(z)}{1 + K\overline{GH}(z)}$$



The system characteristic equation is, then, $1 + K\overline{GH}(z) = 0$ ------------- **

The root locus for this system is a plot of the locus of roots in ** in the z-plane as a function of K. Thus the rules of root-locus construction for discrete-time systems are identical to those for continuous-time systems, since the roots of any equation are dependent only on the coefficients of the equation and are independent of the designation of the variable. Since the rules for root-locus construction are numerous and appear in any standard text for continuous-time control systems , only the most important rules will be repeated here in abbreviated form. These rules are given in Table 3.

Example 4.6

Consider the system with $KG(z) = \dfrac{0.368K(z + 0.717)}{(z - 1)(z - 0.368)}$

Thus the loci originate at $z = 1$ and $z = 0.368$, and terminate at $z = -0.717$ and $z = \infty$. There is one asymptote, at 180°. The breakaway points, obtained from $\dfrac{d}{dz}[G(z)] = 0$

occur at $z = 0.65$ for $K = 0.196$, and $z = -2.08$ for $K = 15.0$. The root locus is then as shown in theFigure . The points of intersection of the root loci with the unit circle may be found by graphical construction, the Jury stability test, or the Routh–Hurwitz criterion. The value of gain for marginal stability (i.e., for the roots to appear on the unit circle) is $K = 2.39$. For this value of gain, the characteristic equation is,

$$z^2 - 0.488z + 1 = 0$$

TABLE 4.3 RULES FOR ROOT-LOCUS CONSTRUCTION

For the characteristic equation

$$1 + K\overline{GH}(z) = 0$$

1. Loci originate on poles of $\overline{GH}(z)$ and terminate on the zeros of $\overline{GH}(z)$.
2. The root locus on the real axis always lies in a section of the real axis to the left of an odd number of poles and zeros on the real axis.
3. The root locus is symmetrical with respect to the real axis.
4. The number of asymptotes is equal to the number of poles of $\overline{GH}(z)$, n_p, minus the number of zeros of $\overline{GH}(z)$, n_z, with angles given by $(2k + 1)\pi/(n_p - n_z)$.
5. The asymptotes intersect the real axis at σ, where

$$\sigma = \frac{\Sigma \text{ poles of } \overline{GH}(z) - \Sigma \text{ zeros of } \overline{GH}(z)}{n_p - n_z}$$

6. The breakaway points are given by the roots of $\dfrac{d[\overline{GH}(z)]}{dz} = 0$

   or, equivalently,

$$D(z)\frac{dN(z)}{dz} - N(z)\frac{dD(z)}{dz} = 0, \qquad \overline{GH}(z) = \frac{N(z)}{D(z)}$$

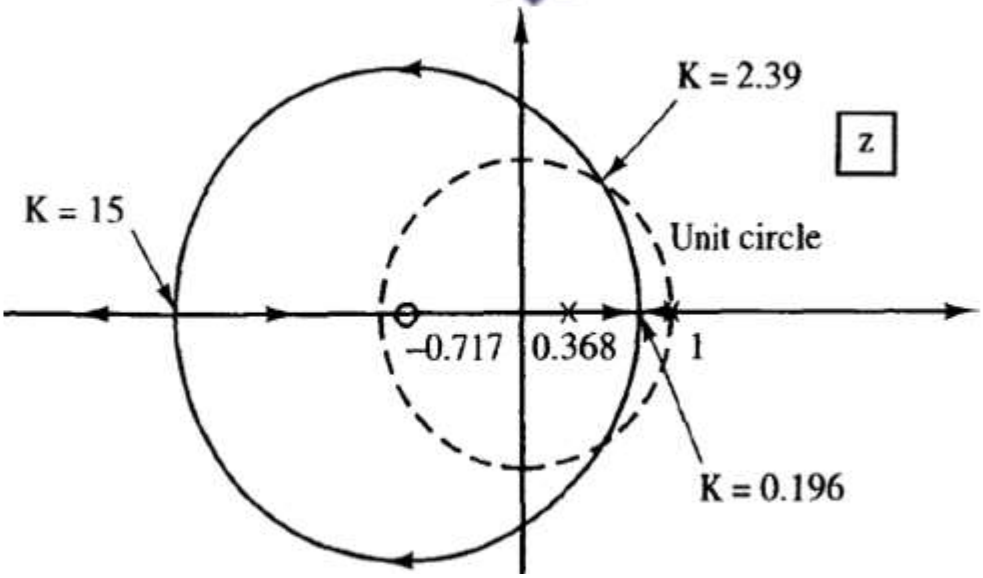
The roots of this equation are

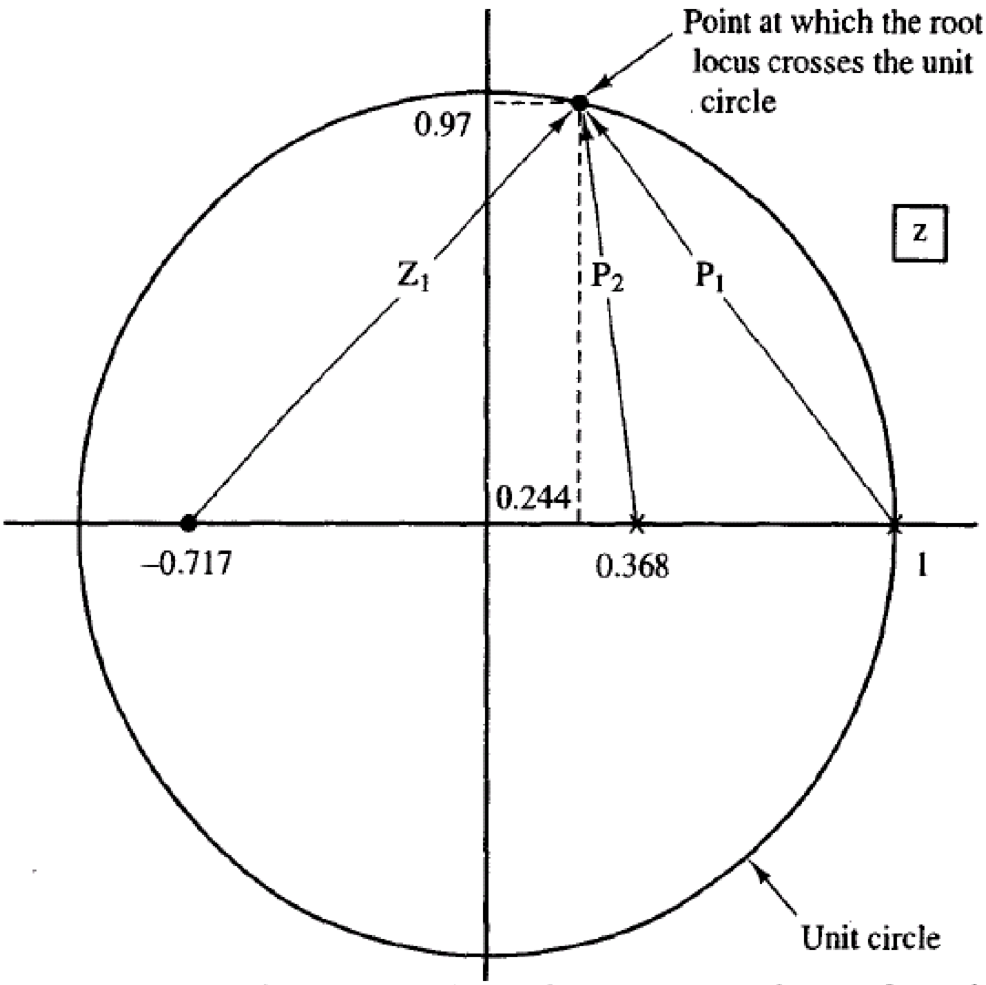
$$z = 0.244 \pm j0.970 = 1\underline{/\pm 75.8^\circ} = 1\underline{/\pm 1.32} \text{ rad} = 1\underline{/\pm \omega T}$$

and thus these are the points at which the root locus crosses the unit circle. Note that the frequency of oscillation for this case is $\omega = 1.32$, since $T = 1$ s.

The value of the gain at points where the root locus crosses the unit circle can also be determined using the root-locus condition that at any point along the locus the magnitude of the open-loop function must be equal to 1 [i.e., $|K\overline{GH}(z)| = 1$]. Using the condition and the Figure , we note that

$$\frac{0.368K(Z_1)}{(P_1)(P_2)} = 1$$





Determination of the system gain at the crossover point on the unit circle.

From Figure the following values can be calculated: $Z_1 = 1.364$, $P_1 = 1.229$, and $P_2 = 0.978$. Using these values in the equation above yields $K = 2.39$. A MATLAB program that solves for and plots a root locus for this example, for $K = 0, 0.1, 0.2, \ldots, 1.0$, is given by

```
format compact
k = 0:0.1:1;
n = [0   0.368   0.264];
d = [1   -1.368   0.368];
r = rlocus(n,d,k);
k
r
pause
plot(real(r),imag(r), 'x')
title('Root Locus')
```

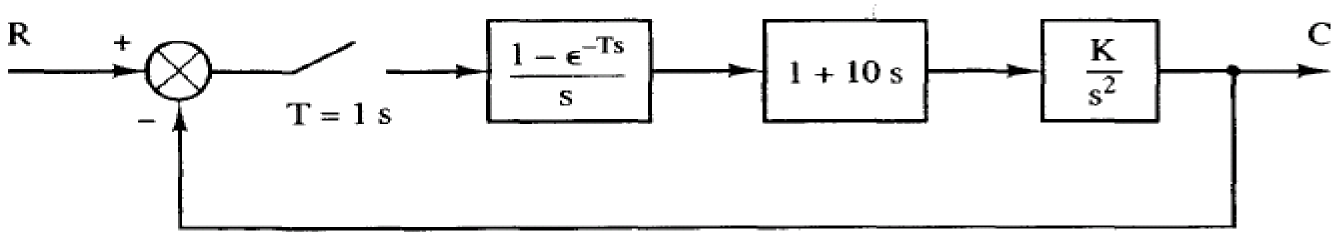
Example 4.7

Consider an open-loop function as

$$KG(s) = \frac{1 - \epsilon^{-sT}}{s}\left[\frac{K(1 + 10s)}{s^2}\right]$$

Applying the z-transform, we obtain

$$KG(z) = \frac{10.5K(z - 0.9048)}{(z - 1)^2}$$



The loci originate at $z = 1$ and terminate at $z = 0.9048$ and $z = -\infty$. There is one asymptote at 180°. The root locus is shown in Figure below. The system becomes unstable when the closed-loop pole leaves the interior of the unit circle at point A shown in the Figure below . The value of K at this point can be determined from the condition $KG(z) = -1$. Therefore,

$$\left.\frac{10.5K(z - 0.9048)}{(z - 1)^2}\right|_{z=-1} = \frac{10.5K(-1.9048)}{4} = -1$$

Thus $K = 0.2$, and we see that the system is stable of $0 < K < 0.2$.