



Discrete State Space Models

In this lecture we will discuss the relation between transfer function and state space model for a discrete time system and various standard or canonical state variable models.

5.1. State Space Model to Transfer Function

Consider a discrete state variable model

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{A}\mathbf{x}(k) + \mathbf{B}u(k) \\ y(k) &= \mathbf{C}\mathbf{x}(k) + \mathbf{D}u(k) \end{aligned} \quad (5.1)$$

Where;

$x(k)$ = n -vector (state vector).

$y(k)$ = m -vector (output vector).

$u(k)$ = r -vector (input vector).

A = $n \times n$ matrix (input matrix).

B = $n \times r$ matrix (state matrix).

C = $m \times n$ matrix (output matrix).

D = $m \times r$ matrix (Direct transmission matrix).

Taking the Z-transform on both sides of Eqn. (5.1), we get

$$\begin{aligned} zX(z) - z\mathbf{x}_0 &= \mathbf{A}X(z) + \mathbf{B}U(z) \\ Y(z) &= \mathbf{C}X(z) + \mathbf{D}U(z) \end{aligned}$$

where x_0 is the initial state of the system.

$$\begin{aligned} \Rightarrow & (zI - \mathbf{A})X(z) = z\mathbf{x}_0 + \mathbf{B}U(z) \\ \text{or, } X(z) &= (zI - \mathbf{A})^{-1}z\mathbf{x}_0 + (zI - \mathbf{A})^{-1}\mathbf{B}U(z) \end{aligned}$$



To find out the transfer function, we assume that the initial conditions are zero, i.e., $x_0 = 0$, thus

$$Y(z) = (C(zI - A)^{-1}B + D)U(z)$$

Therefore, the transfer function becomes

$$\boxed{G(z) = \frac{Y(z)}{U(z)} = C(zI - A)^{-1}B + D} \quad (5.2)$$

which has the same form as that of a continuous time system.

5.2. Various Canonical Forms

We have seen that transform domain analysis of a digital control system yields a transfer function of the following form.

$$G(z) = \frac{Y(z)}{U(z)} = \frac{\beta_0 z^m + \beta_1 z^{m-1} + \dots + \beta_m}{z^n + \alpha_1 z^{n-1} + \dots + \alpha_n} \quad m \leq n \quad (5.3)$$

Various canonical state variable models can be derived from the above transfer function model.

5.2.1 Controllable canonical form

Consider the transfer function as given in Eqn. (5.3). Without loss of generality, let us consider the case when $m = n$. Let

$$\frac{X(z)}{U(z)} = \frac{1}{z^n + \alpha_1 z^{n-1} + \dots + \alpha_n}$$

In time domain, the above equation may be written as



$$\bar{x}(k+n) + \alpha_1 \bar{x}(k+n-1) + \dots + \alpha_n \bar{x}(k) = u(k)$$

Now, the output $Y(z)$ may be written in terms of $\bar{X}(z)$ as

$$Y(z) = (\beta_0 z^n + \beta_1 z^{n-1} + \dots + \beta_n) \bar{X}(z)$$

or in time domain as

$$y(k) = \beta_0 \bar{x}(k+n) + \beta_1 \bar{x}(k+n-1) + \dots + \beta_n \bar{x}(k)$$

The block diagram representation of above equations is shown in Figure 1. State variables are selected as shown in Figure 1. The state equations are then written as:

$$x_1(k+1) = x_2(k)$$

$$x_2(k+1) = x_3(k)$$

$$\vdots = \vdots$$

$$x_n(k+1) = -\alpha_n x_1(k) - \alpha_{n-1} x_2(k) - \dots - \alpha_1 x_n(k) + u(k)$$

Output equation can be written as by following the Figure 1.

$$y(k) = (\beta_n - \alpha_n \beta_0) x_1(k) + (\beta_{n-1} - \alpha_{n-1} \beta_0) x_2(k) + \dots + (\beta_1 - \alpha_1 \beta_0) x_n(k) + \beta_0 u(k)$$

In state space form, we have;

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}u(k)$$

$$y(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}u(k)$$

(5.4)

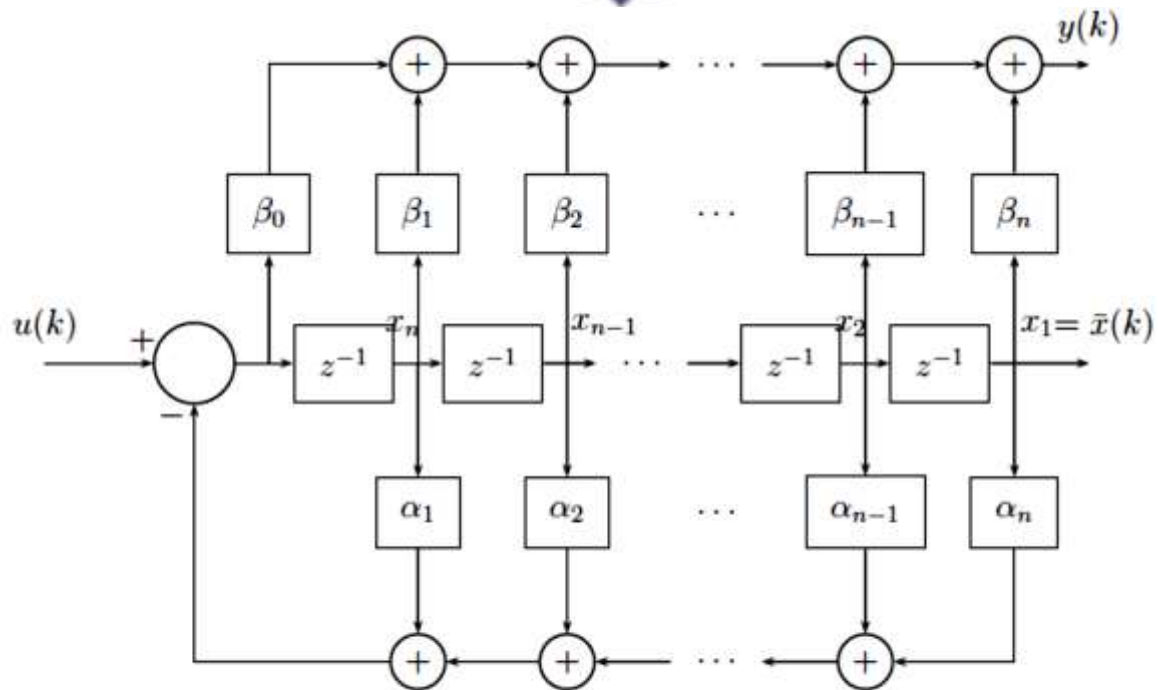


Figure 1: Block Diagram representation of controllable canonical form

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -\alpha_n & -\alpha_{n-1} & -\alpha_{n-2} & \dots & -\alpha_1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$C = [\beta_n - \alpha_n\beta_0 \quad \beta_{n-1} - \alpha_{n-1}\beta_0 \quad \dots \quad \beta_1 - \alpha_1\beta_0] \quad D = \beta_0$$

Example 5.1:

Find the Controllable canonical form of;

$$\frac{Y(z)}{U(z)} = \frac{z + 1}{z^2 + 1.3z + 0.4}$$

**Solution:**

Analysing the coefficient as;

$\beta_0=0, \beta_1=1, \beta_2=1$, and $\alpha_1=1.3, \alpha_2=0.4$, then ;

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.4 & -1.3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = [1 \quad 1] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

5.2.2 Observable Canonical Form

Equation (3) may be rewritten as

$$(z^n + \alpha_1 z^{n-1} + \dots + \alpha_n) Y(z) = (\beta_0 z^n + \beta_1 z^{n-1} + \dots + \beta_n) U(z)$$

$$\text{or, } z^n[Y(z) - \beta_0 U(z)] + z^{n-1}[\alpha_1 Y(z) - \beta_1 U(z)] + \dots + [\alpha_n Y(z) - \beta_n U(z)] = 0$$

$$\text{or, } Y(z) = \beta_0 U(z) - z^{-1}[\alpha_1 Y(z) - \beta_1 U(z)] - \dots - z^{-n}[\alpha_n Y(z) - \beta_n U(z)]$$

The corresponding block diagram is shown in Figure 2. Choosing the outputs of the delay blocks

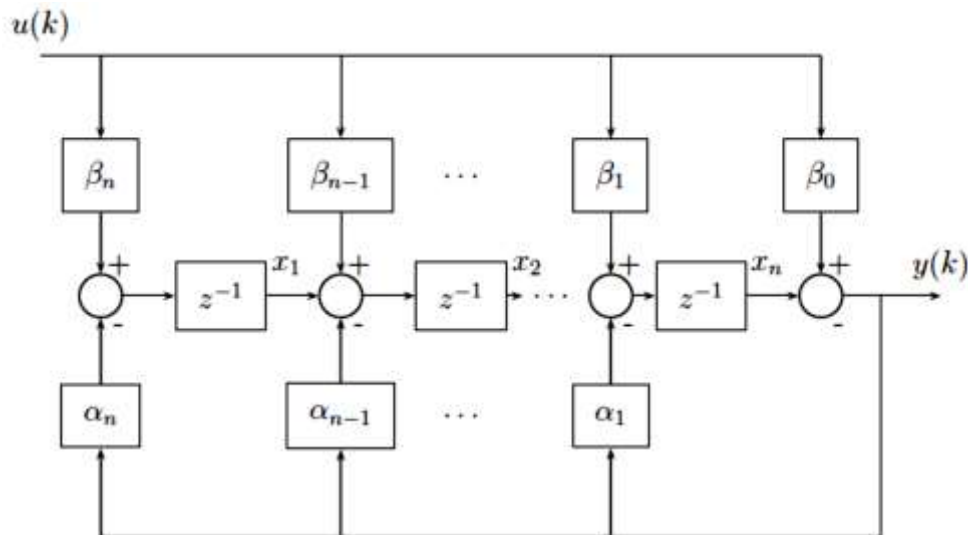


Figure 2: Block Diagram representation of observable canonical form



as the state variables, we have following state equations

$$\begin{aligned}
 x_n(k+1) &= x_{n-1}(k) - \alpha_1(x_n(k) + \beta_0 u(k)) + \beta_1 u(k) \\
 x_{n-1}(k+1) &= x_{n-2}(k) - \alpha_2(x_n(k) + \beta_0 u(k)) + \beta_2 u(k) \\
 &\vdots = \vdots \\
 x_1(k+1) &= -\alpha_n(x_n(k) + \beta_0 u(k)) + \beta_n u(k) \\
 y(k) &= x_n(k) + \beta_0 u(k)
 \end{aligned}$$

This can be rewritten in matrix form (eq.4) with

$$A = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & -\alpha_n \\ 1 & 0 & 0 & \dots & 0 & -\alpha_{n-1} \\ 0 & 1 & 0 & \dots & 0 & -\alpha_{n-2} \\ \vdots & \vdots & \vdots & \dots & \vdots & \\ 0 & 0 & 0 & \dots & 1 & -\alpha_1 \end{bmatrix} \quad B = \begin{bmatrix} \beta_n - \alpha_n \beta_0 \\ \beta_{n-1} - \alpha_{n-1} \beta_0 \\ \vdots \\ \beta_1 - \alpha_1 \beta_0 \end{bmatrix} \quad C = [0 \ 0 \ \dots \ 1] \quad D = \beta_0$$

Example 5.2:

Find the Observable canonical form of;

$$\frac{Y(z)}{U(z)} = \frac{z + 1}{z^2 + 1.3z + 0.4}$$

Solution:

Analysing the coefficient as;

$\beta_0=0$, $\beta_1=1$, $\beta_2=1$, and $\alpha_1=1.3$, $\alpha_2=0.4$, then ;

$$\begin{aligned}
 \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} &= \begin{bmatrix} 0 & -0.4 \\ 1 & -1.3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k) \\
 y(k) &= [0 \ 1] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}
 \end{aligned}$$



5.2.3 Duality

In previous two sections we observed that the system matrix A in observable canonical form is transpose of the system matrix in controllable canonical form. Similarly, control matrix B in observable canonical form is transpose of output matrix C in controllable canonical form. So also output matrix C in observable canonical form is transpose of control matrix B in controllable canonical form.

5.2.4 Jordan Canonical Form

In Jordan canonical form, the system matrix A represents a diagonal matrix for distinct poles which basically form the diagonal elements of A.

Assume that $z = \lambda_i, i = 1, 2, \dots, n$ are the distinct poles of the given transfer function (5.3). Then partial fraction expansion of the transfer function yields;

$$\begin{aligned}
 \frac{Y(z)}{U(z)} &= \beta_0 + \frac{\bar{\beta}_1 z^{n-1} + \bar{\beta}_2 z^{n-2} + \dots + \bar{\beta}_n}{z^n + \alpha_1 z^{n-1} + \dots + \alpha_n} \\
 &= \beta_0 + \frac{\bar{\beta}_1 z^{n-1} + \bar{\beta}_2 z^{n-2} + \dots + \bar{\beta}_n}{(z - \lambda_1)(z - \lambda_2) \dots (z - \lambda_n)} \\
 &= \beta_0 + \frac{r_1}{z - \lambda_1} + \frac{r_2}{z - \lambda_2} + \dots + \frac{r_n}{z - \lambda_n}
 \end{aligned} \tag{5.5}$$

A parallel realization of the transfer function (5.5) is shown in Figure 3.

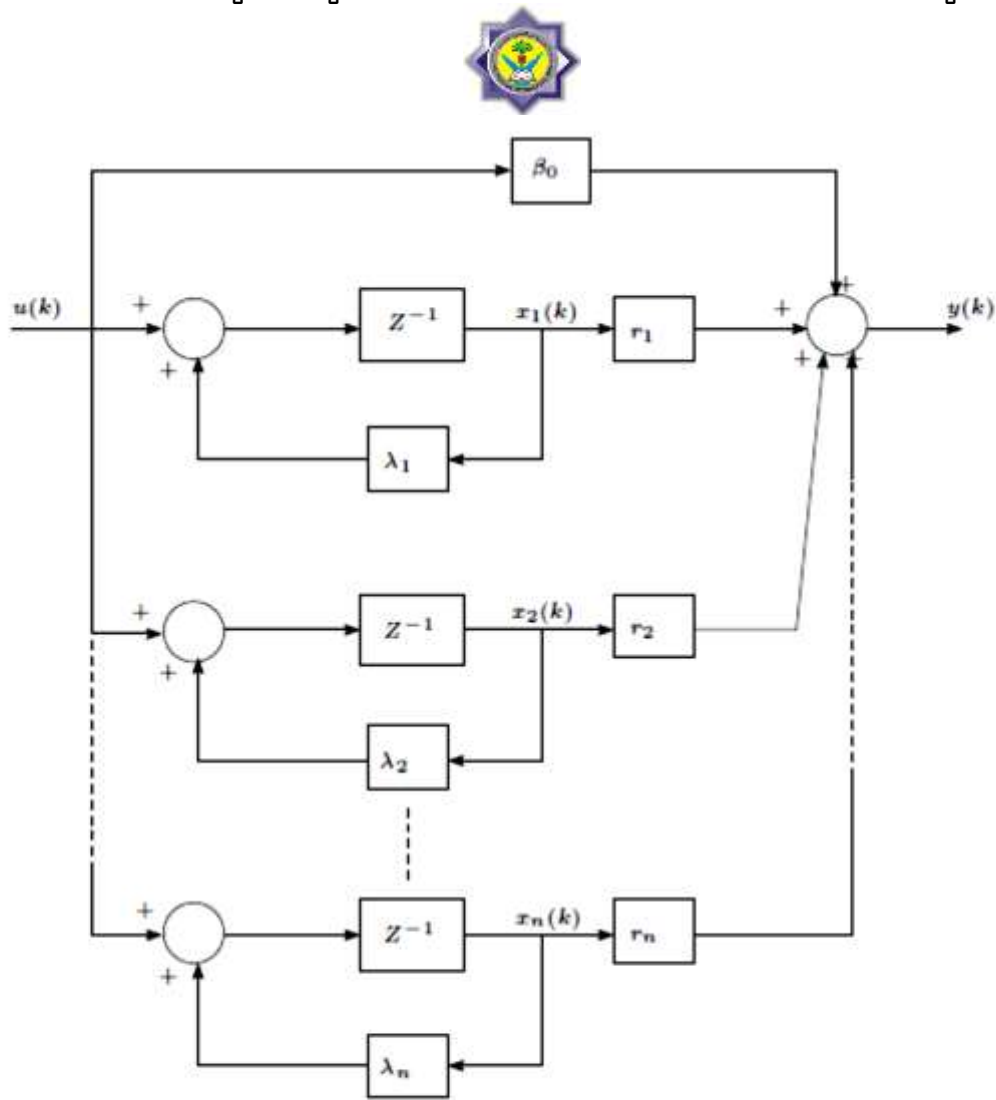


Figure 3: Block Diagram representation of Jordan canonical form

Considering the outputs of the delay blocks as the state variables, we can construct the state model in matrix form (5.4), with

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_n \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \quad C = [r_1 \ r_2 \ \dots \ r_n] \quad D = \beta_0$$



When the matrix A has repeated eigenvalues, it cannot be expressed in a proper diagonal form. However, it can be expressed in a Jordan canonical form which is nearly a diagonal matrix. Let us consider that the system has eigenvalues, $\lambda_1, \lambda_1, \lambda_2$ and λ_3 . In that case, A matrix in Jordan canonical form will be

$$A = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{bmatrix}$$

1. The diagonal elements of the matrix A are eigenvalues of the same.
2. The elements below the principal diagonal are zero.
3. Some of the elements just above the principal diagonal are one.
4. The matrix can be divided into a number of blocks, called Jordan blocks, along the diagonal. Each block depends on the multiplicity of the eigenvalue associated with it. For example Jordan block associated with a eigenvalue z_1 of multiplicity 4 can be written as;

$$A = \begin{bmatrix} z_1 & 1 & 0 & 0 \\ 0 & z_1 & 1 & 0 \\ 0 & 0 & z_1 & 1 \\ 0 & 0 & 0 & z_1 \end{bmatrix}$$

Example 5.3:

Find the Jordan canonical form of;

$$\frac{Y(z)}{U(z)} = \frac{z + 1}{z^2 + 1.3z + 0.4}$$

Solution:

Using partial fraction method the transfer function may be rewritten as;



$$\frac{Y(z)}{U(z)} = \frac{5/3}{z + 0.5} + \frac{-2/3}{z + 0.8}$$

Then $\lambda_1 = -0.5$, $\lambda_2 = -0.8$, $r_1 = 5/3$, $\beta_0 = 0$, and $r_2 = -2/3$, then

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} -0.5 & 0 \\ 0 & -0.8 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} \frac{5}{3} & \frac{-2}{3} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Example 5.4: Consider the following discrete transfer function.

$$G(z) = \frac{0.17z + 0.04}{z^2 - 1.1z + 0.24}$$

Find out the state variable model in 3 different canonical forms.

Solution:

The state variable model in controllable canonical form can directly be derived from the transfer function, where the A, B, C and D matrices are as follows:

$$A = \begin{bmatrix} 0 & 1 \\ -0.24 & 1.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [0.04 \quad 0.17], \quad D = 0$$

The matrices in state model corresponding to observable canonical form are obtained as,

$$A = \begin{bmatrix} 0 & -0.24 \\ 1 & 1.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.04 \\ 0.17 \end{bmatrix}, \quad C = [0 \quad 1], \quad D = 0$$

To find out the state model in Jordan canonical form, we need to fact expand the transfer function using partial fraction, as



$$G(z) = \frac{0.17z + 0.04}{z^2 - 1.1z + 0.24} = \frac{0.352}{z - 0.8} + \frac{-0.182}{z - 0.3}$$

Thus the A, B, C and D matrices will be:

$$A = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = [0.352 \quad -0.182], \quad D = 0$$

Example 5.5: Consider the following discrete transfer function.

$$\frac{Y(z)}{U(z)} = \frac{z + 1}{z^2 + 1.3z + 0.4}$$

Find out the state variable model in different canonical forms.

Controllable Canonical Form:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.4 & -1.3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = [1 \quad 1] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Observable Canonical Form:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & -0.4 \\ 1 & -1.3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = [0 \quad 1] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Jordan (Diagonal) Canonical Form:

The given pulse transfer function $Y(z)/U(z)$ can be expanded as follows:

$$\frac{Y(z)}{U(z)} = \frac{5/3}{z + 0.5} + \frac{-2/3}{z + 0.8}$$

Hence;

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} -0.5 & 0 \\ 0 & -1.3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$



$$y(k) = [0 \quad 1] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

5.3. State Space to Transfer Function

To retrieve the discrete domain transfer function $G(z)$ from a given state space matrices (A , B , C , and D), equation (5.2) is recalled as following;

$$G(z) = \frac{Y(z)}{U(z)} = C(zI - A)^{-1}B + D$$

where $(zI - A)^{-1}$ may be found as;

$$(zI - A)^{-1} = \frac{\text{adj}(zI - A)}{\det(zI - A)}$$

Therefore, the transfer function may be written as;

$$\boxed{G(z) = \frac{C \text{adj}(zI - A)B + D \det(zI - A)}{\det(zI - A)}} \quad (5.6)$$

Example 5.6:

Find the transfer function corresponding to the state space matrices below;

$$A = \begin{bmatrix} 0 & 1 \\ -0.4 & -1.3 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = [1 \quad 1], \text{ and } D=0.$$

**Solution:**

Finding first $(zI-A)$ as follows;

$$\left(z \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -0.4 & -1.3 \end{bmatrix} \right) = \left(\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -0.4 & -1.3 \end{bmatrix} \right) = \begin{bmatrix} z & -1 \\ 0.4 & z + 1.3 \end{bmatrix}$$

and by applying equation 5.6, the transfer function may be found as;

$$G(z) = \frac{\begin{bmatrix} 1 & 1 \end{bmatrix} \text{adj} \begin{bmatrix} z & -1 \\ 0.4 & z + 1.3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}}{\det \begin{bmatrix} z & -1 \\ 0.4 & z + 1.3 \end{bmatrix}} = \frac{\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} z + 1.3 & 1 \\ -0.4 & z \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}}{z^2 + 1.3z + 0.4}$$

$$G(z) = \frac{\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ z \end{bmatrix}}{z^2 + 1.3z + 0.4} = \frac{1 + z}{z^2 + 1.3z + 0.4}$$

Example 5.7:

Find the transfer function corresponding to the state space matrices below;

$$A = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.3 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, C = [0.352 \quad -0.182], \text{ and } D=0.$$

Finding first $(zI-A)$ as follows;

$$\left(\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 0.8 & 0 \\ 0 & 0.3 \end{bmatrix} \right) = \begin{bmatrix} z - 0.8 & 0 \\ 0 & z - 0.3 \end{bmatrix}$$

and by applying equation 5.6, the transfer function may be found as;

$$G(z) = \frac{\begin{bmatrix} 0.352 & -0.182 \end{bmatrix} \text{adj} \begin{bmatrix} z - 0.8 & 0 \\ 0 & z - 0.3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\det \begin{bmatrix} z - 0.8 & 0 \\ 0 & z - 0.3 \end{bmatrix}} = \frac{\begin{bmatrix} 0.352 & -0.182 \end{bmatrix} \begin{bmatrix} z - 0.3 & 0 \\ 0 & z - 0.8 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}{(z - 0.8)(z - 0.3)}$$

$$G(z) = \frac{\begin{bmatrix} 0.352 & -0.182 \end{bmatrix} \begin{bmatrix} z - 0.3 \\ z - 0.8 \end{bmatrix}}{(z - 0.8)(z - 0.3)} = \frac{0.352(z - 0.3) - 0.182(z - 0.8)}{(z - 0.8)(z - 0.3)}$$

$$G(z) = \frac{0.17z + 0.04}{(z - 0.8)(z - 0.3)}$$