Fourier Series :

Fourier series are the basic tool for representing periodic functions, which play an important role in applications. A function $f(x)$ is called a **periodic function** if $f(x)$ is defined for all real x (perhaps except at some points, such as $x = \pm\pi/2, \pm 3\pi/2, \cdots$ for $\tan x$) and if there is some positive number p, called a **period** of $f(x)$, such that

$$f(x + p) = f(x) \qquad \text{for all } x. \tag{1}$$

The graph of such a function is obtained by periodic repetition of its graph in any interval of length p (Fig. 255).

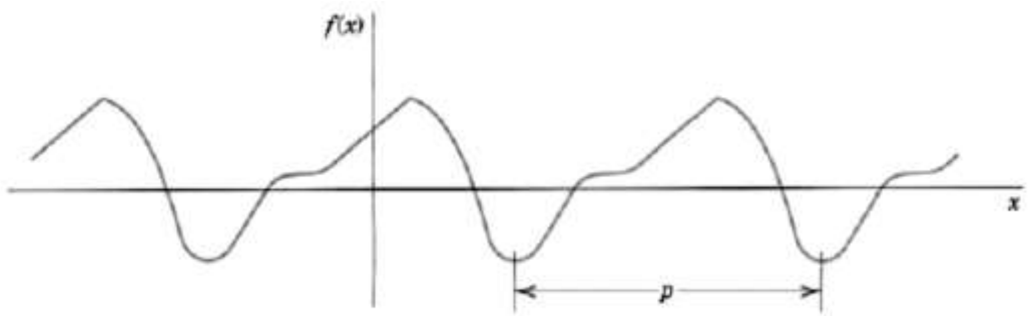


Fig. 255. **Periodic function**

Familiar periodic functions are the cosine and sine functions. Examples of functions that are not periodic are x, x^2, x^3, e^x, $\cosh x$, and $\ln x$, to mention just a few.

If $f(x)$ has period p, it also has the period $2p$ because (1) implies $f(x + 2p) = f([x + p] + p) = f(x + p) = f(x)$, etc.; thus for any integer $n = 1, 2, 3, \cdots$,

$$f(x + np) = f(x) \qquad \text{for all } x. \tag{2}$$

Furthermore if $f(x)$ and $g(x)$ have period p, then $af(x) + bg(x)$ with any constants a and b also has the period p.

Our problem in the first few sections of this chapter will be the representation of various ***functions*** $\boldsymbol{f(x)}$ ***of period*** $\boldsymbol{2\pi}$ in terms of the simple functions

$$1, \quad \cos x, \quad \sin x, \quad \cos 2x, \quad \sin 2x, \cdots, \quad \cos nx, \quad \sin nx, \cdots. \tag{3}$$

All these functions have the period 2π. They form the so-called **trigonometric system.** Figure 256 shows the first few of them (except for the constant 1, which is periodic with any period).

The series to be obtained will be a **trigonometric series,** that is, a series of the form

$$\begin{aligned} &a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots \\ &= a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \end{aligned} \tag{4}$$

$a_0, a_1, b_1, a_2, b_2, \cdots$ are constants, called the **coefficients** of the series. We see that each term has the period 2π. Hence *if the coefficients are such that the series converges, its sum will be a function of period* 2π.

It can be shown that if the series on the left side of (4) converges, then inserting parentheses on the right gives a series that converges and has the same sum as the series on the left. This justifies the equality in (4).

Now suppose that $f(x)$ is a given function of period 2π and is such that it can be **represented** by a series (4), that is, (4) converges and, moreover, has the sum $f(x)$. Then, using the equality sign, we write

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{5}$$

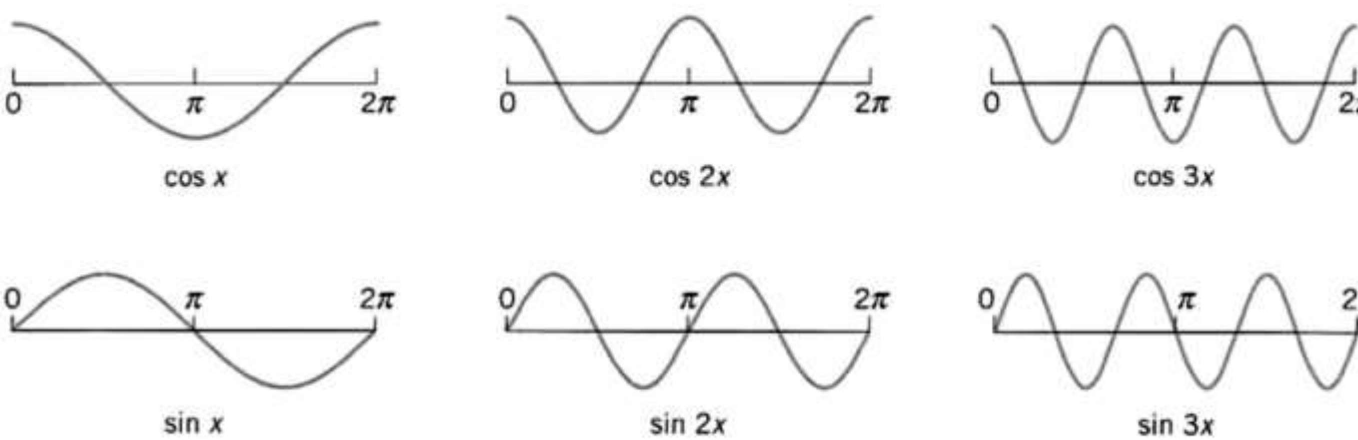


Fig. 256. Cosine and sine functions having the period 2π

and call (5) the **Fourier series** of $f(x)$. We shall prove that in this case the coefficients of (5) are the so-called **Fourier coefficients** of $f(x)$, given by the **Euler formulas**

(6)

(a) $$a_0 = \frac{1}{p} \int_{-\pi}^{\pi} f(x)\, dx$$

(b) $$a_n = \frac{1}{\frac{p}{2}} \int_{-\pi}^{\pi} f(x) \cos n \frac{2\pi}{p} x\, dx \qquad n = 1, 2, \cdots$$

(c) $$b_n = \frac{1}{\frac{p}{2}} \int_{-\pi}^{\pi} f(x) \sin n \frac{2\pi}{p} x\, dx \qquad n = 1, 2, \cdots.$$

Example 1:

Periodic Rectangular Wave (Fig. 257a)

Find the Fourier coefficients of the periodic function $f(x)$ in Fig. 257a. The formula is

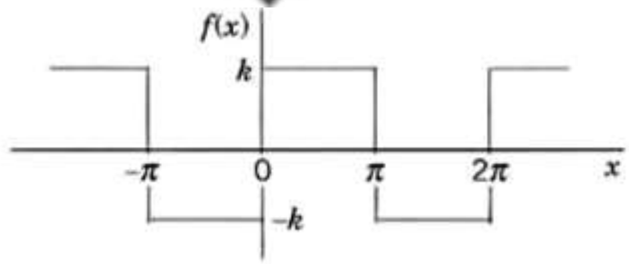
(7) $$f(x) = \begin{cases} -k & \text{if} \quad -\pi < x < 0 \\ k & \text{if} \quad 0 < x < \pi \end{cases} \qquad \text{and} \qquad f(x + 2\pi) = f(x).$$

Functions of this kind occur as external forces acting on mechanical systems, electromotive forces in electric circuits, etc. (The value of $f(x)$ at a single point does not affect the integral; hence we can leave $f(x)$ undefined at $x = 0$ and $x = \pm\pi$.)

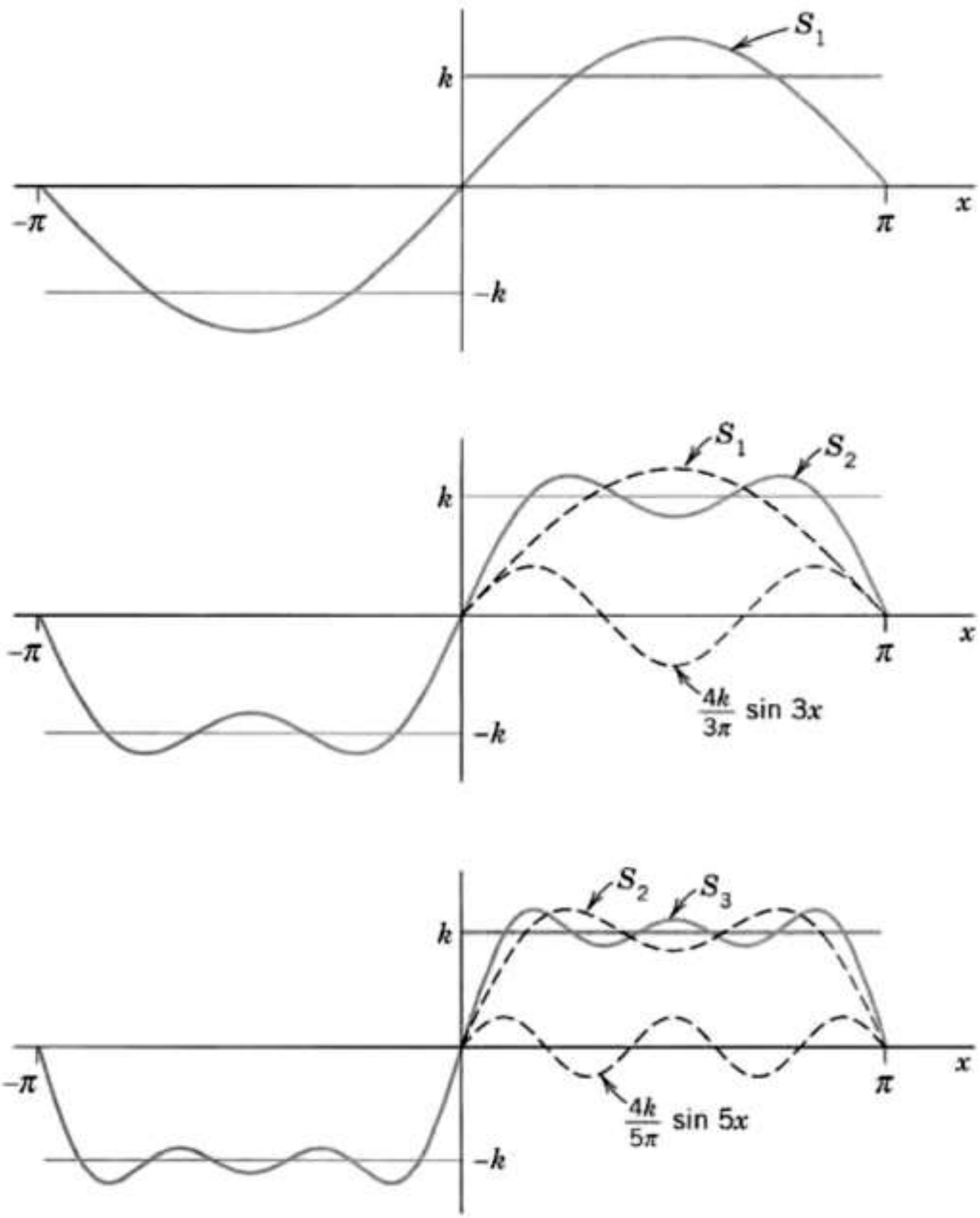
Solution. From (6a) we obtain $a_0 = 0$. This can also be seen without integration, since the area under the curve of $f(x)$ between $-\pi$ and π is zero. From (6b),

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx = \frac{1}{\pi} \left[\int_{-\pi}^{0} (-k) \cos nx\, dx + \int_{0}^{\pi} k \cos nx\, dx \right]$$

$$= \frac{1}{\pi} \left[-k \frac{\sin nx}{n} \Bigg|_{-\pi}^{0} + k \frac{\sin nx}{n} \Bigg|_{0}^{\pi} \right] = 0$$



(a) The given function $f(x)$ (Periodic rectangular wave)



(b) The first three partial sums of the corresponding Fourier series

Fig. 257. **Eample 1**

because $\sin nx = 0$ at $-\pi$, 0, and π for all $n = 1, 2, \cdots$. Similarly, from (6c) we obtain

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\sin nx\,dx + \int_{0}^{\pi} k\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[k\,\frac{\cos nx}{n}\bigg|_{-\pi}^{0} - k\,\frac{\cos nx}{n}\bigg|_{0}^{\pi}\right].$$

Since $\cos(-\alpha) = \cos\alpha$ and $\cos 0 = 1$, this yields

$$b_n = \frac{k}{n\pi}\left[\cos 0 - \cos(-n\pi) - \cos n\pi + \cos 0\right] = \frac{2k}{n\pi}(1 - \cos n\pi).$$

Now, $\cos\pi = -1$, $\cos 2\pi = 1$, $\cos 3\pi = -1$, etc.; in general,

$$\cos n\pi = \begin{cases} -1 & \text{for odd } n, \\ 1 & \text{for even } n, \end{cases} \qquad \text{and thus} \qquad 1 - \cos n\pi = \begin{cases} 2 & \text{for odd } n, \\ 0 & \text{for even } n. \end{cases}$$

Hence the Fourier coefficients b_n of our function are

$$b_1 = \frac{4k}{\pi}, \qquad b_2 = 0, \qquad b_3 = \frac{4k}{3\pi}, \qquad b_4 = 0, \qquad b_5 = \frac{4k}{5\pi}, \cdots.$$

Since the a_n are zero, the Fourier series of $f(x)$ is

(8) $$\frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right).$$

The partial sums are

$$S_1 = \frac{4k}{\pi}\sin x, \qquad S_2 = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x\right), \qquad \text{etc.,}$$

Furthermore, assuming that $f(x)$ is the sum of the series and setting $x = \pi/2$, we have

$$f\left(\frac{\pi}{2}\right) = k = \frac{4k}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - + \cdots\right).$$

thus

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots = \frac{\pi}{4}.$$

Derivation of the Euler Formulas:

Theorem 1:

Orthogonality of the Trigonometric System (3)

The trigonometric system (3) *is orthogonal on the interval* $-\pi \leqq x \leqq \pi$ (hence also on $0 \leqq x \leqq 2\pi$ or any other interval of length 2π because of periodicity); *that is, the integral of the product of any two functions in* (3) *over that interval is* 0, *so that for any integers n and m,*

(9)

$$\text{(a)} \quad \int_{-\pi}^{\pi} \cos nx \cos mx \, dx = 0 \qquad (n \neq m)$$

$$\text{(b)} \quad \int_{-\pi}^{\pi} \sin nx \sin mx \, dx = 0 \qquad (n \neq m)$$

$$\text{(c)} \quad \int_{-\pi}^{\pi} \sin nx \cos mx \, dx = 0 \qquad (n \neq m \text{ or } n = m).$$

Proof:

This follows simply by transforming the integrands trigonometrically from products into sums. In (9a) and (9b), by (11) in App. A3.1,

$$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos (n + m)x \, dx + \frac{1}{2}\int_{-\pi}^{\pi} \cos (n - m)x \, dx$$

$$\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos (n - m)x \, dx - \frac{1}{2}\int_{-\pi}^{\pi} \cos (n + m)x \, dx.$$

$$\int_{-\pi}^{\pi} \sin nx \cos mx \, dx = \frac{1}{2}\int_{-\pi}^{\pi} \sin (n + m)x \, dx + \frac{1}{2}\int_{-\pi}^{\pi} \sin (n - m)x \, dx = 0 + 0.$$

Example 2:

Periodic Rectangular Wave

Find the Fourier series of the function (Fig. 259)

$$f(x) = \begin{cases} 0 & \text{if } -2 < x < -1 \\ k & \text{if } -1 < x < 1 \\ 0 & \text{if } 1 < x < 2 \end{cases} \qquad p = 2L = 4, \quad L = 2.$$

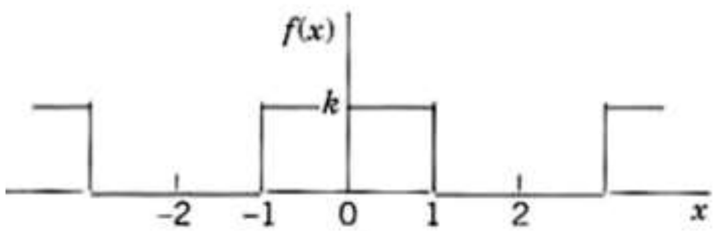


Fig. 259. Example 1

Solution. From (6a) we obtain $a_0 = k/2$ (verify!). From (6b) we obtain

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx = \frac{1}{2}\int_{-1}^{1} k\cos\frac{n\pi x}{2}\,dx = \frac{2k}{n\pi}\sin\frac{n\pi}{2}.$$

Thus $a_n = 0$ if n is even and

$$a_n = 2k/n\pi \quad \text{if} \quad n = 1, 5, 9, \cdots, \qquad a_n = -2k/n\pi \quad \text{if} \quad n = 3, 7, 11, \cdots.$$

From (6c) we find that $b_n = 0$ for $n = 1, 2, \cdots$. Hence the Fourier series is

$$f(x) = \frac{k}{2} + \frac{2k}{\pi}\left(\cos\frac{\pi}{2}x - \frac{1}{3}\cos\frac{3\pi}{2}x + \frac{1}{5}\cos\frac{5\pi}{2}x - + \cdots\right).$$

Example 3:

Periodic Rectangular Wave

Find the Fourier series of the function (Fig. 260)

$$f(x) = \begin{cases} -k & \text{if} \quad -2 < x < 0 \\ k & \text{if} \quad 0 < x < 2 \end{cases} \qquad p = 2L = 4, \qquad L = 2.$$

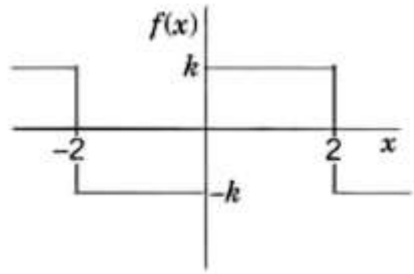


Fig. 260. **Example 2**

Solution. $a_0 = 0$ from (6a). From (6b), with $1/L = 1/2$,

$$a_n = \frac{1}{2}\left[\int_{-2}^{0}(-k)\cos\frac{n\pi x}{2}\,dx + \int_{0}^{2}k\cos\frac{n\pi x}{2}\,dx\right]$$
$$= \frac{1}{2}\left[-\frac{2k}{n\pi}\sin\frac{n\pi x}{2}\Big|_{-2}^{0} + \frac{2k}{n\pi}\sin\frac{n\pi x}{2}\Big|_{0}^{2}\right] = 0,$$

so that the Fourier series has no cosine terms. From (6c),

$$b_n = \frac{1}{2}\left[\frac{2k}{n\pi}\cos\frac{n\pi x}{2}\Big|_{-2}^{0} - \frac{2k}{n\pi}\cos\frac{n\pi x}{2}\Big|_{0}^{2}\right]$$
$$= \frac{k}{n\pi}(1 - \cos n\pi - \cos n\pi + 1) = \begin{cases} 4k/n\pi & \text{if} \quad n = 1, 3, \cdots \\ 0 & \text{if} \quad n = 2, 4, \cdots. \end{cases}$$

Hence the Fourier series of $f(x)$ is

$$f(x) = \frac{4k}{\pi}\left(\sin\frac{\pi}{2}x + \frac{1}{3}\sin\frac{3\pi}{2}x + \frac{1}{5}\sin\frac{5\pi}{2}x + \cdots\right).$$

Example 3:

Half-Wave Rectifier

A sinusoidal voltage $E \sin \omega t$, where t is time, is passed through a half-wave rectifier that clips the negative portion of the wave (Fig. 261). Find the Fourier series of the resulting periodic function

$$u(t) = \begin{cases} 0 & \text{if} \quad -L < t < 0, \\ E \sin \omega t & \text{if} \quad 0 < t < L \end{cases} \qquad p = 2L = \frac{2\pi}{\omega}, \qquad L = \frac{\pi}{\omega}.$$

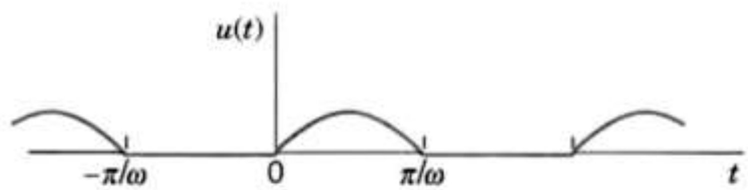


Fig. 261. Half-wave rectifier

Solution. Since $u = 0$ when $-L < t < 0$, we obtain from (6a), with t instead of x,

$$a_0 = \frac{\omega}{2\pi} \int_0^{\pi/\omega} E \sin \omega t \, dt = \frac{E}{\pi}$$

and from (6b), by using formula (11) in App. A3.1 with $x = \omega t$ and $y = n\omega t$,

$$a_n = \frac{\omega}{\pi} \int_0^{\pi/\omega} E \sin \omega t \cos n\omega t \, dt = \frac{\omega E}{2\pi} \int_0^{\pi/\omega} [\sin (1+n)\omega t + \sin (1-n)\omega t] \, dt.$$

If $n = 1$, the integral on the right is zero, and if $n = 2, 3, \cdots$, we readily obtain

$$a_n = \frac{\omega E}{2\pi} \left[-\frac{\cos (1+n)\omega t}{(1+n)\omega} - \frac{\cos (1-n)\omega t}{(1-n)\omega} \right]_0^{\pi/\omega}$$

$$= \frac{E}{2\pi} \left(\frac{-\cos (1+n)\pi + 1}{1+n} + \frac{-\cos (1-n)\pi + 1}{1-n} \right).$$

If n is odd, this is equal to zero, and for even n we have

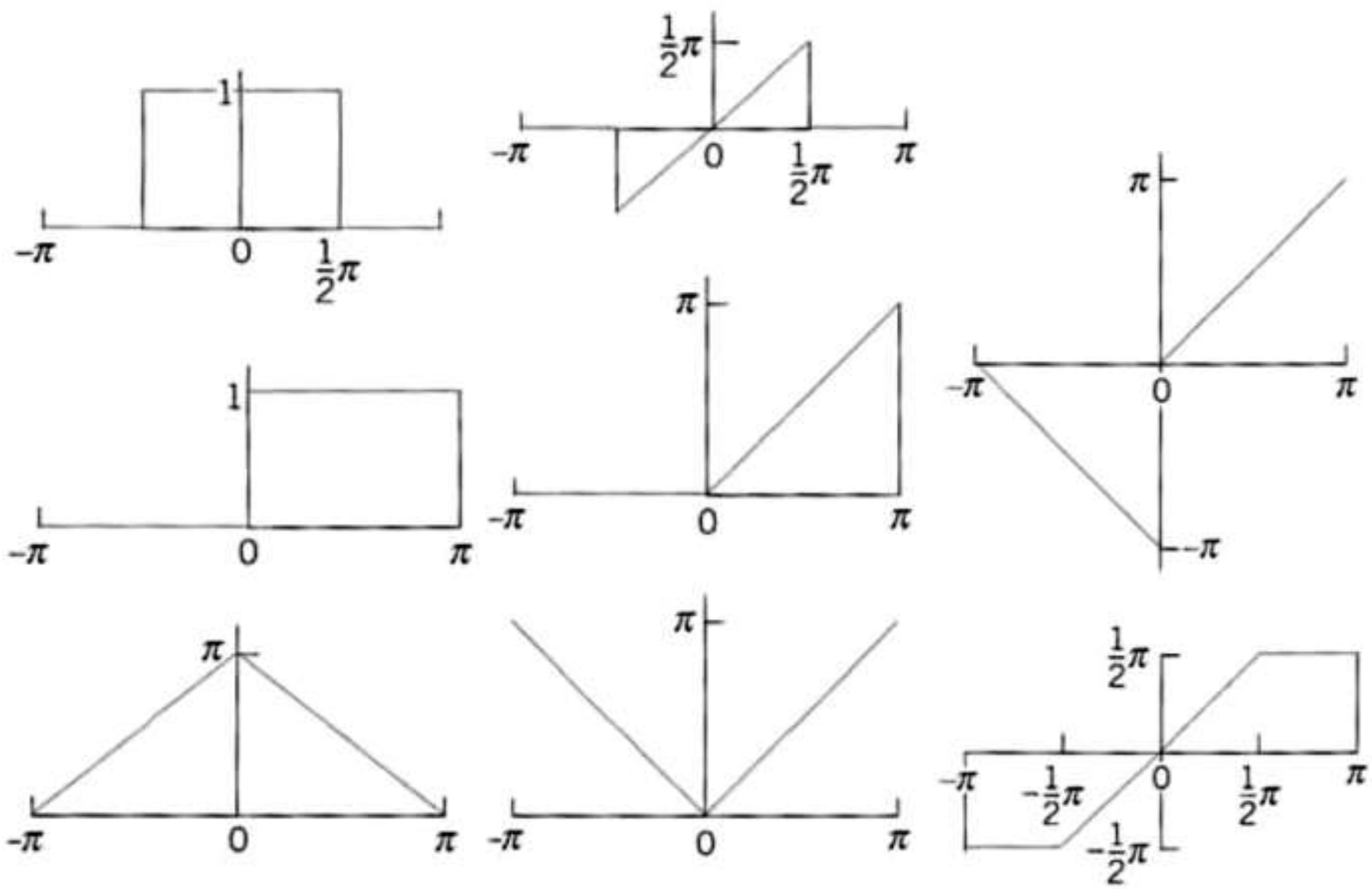
$$a_n = \frac{E}{2\pi}\left(\frac{2}{1+n} + \frac{2}{1-n}\right) = -\frac{2E}{(n-1)(n+1)\pi} \qquad (n = 2, 4, \cdots).$$

In a similar fashion we find from (6c) that $b_1 = E/2$ and $b_n = 0$ for $n = 2, 3, \cdots$. Consequently,

$$u(t) = \frac{E}{\pi} + \frac{E}{2}\sin\omega t - \frac{2E}{\pi}\left(\frac{1}{1\cdot 3}\cos 2\omega t + \frac{1}{3\cdot 5}\cos 4\omega t + \cdots\right). \qquad ■$$

<u>Exercises of Fourier Series</u>

Showing the details of your work, find the Fourier series of the given f(x), which is assumed to have period 2π.



$$f(x) = x^2 \quad (-\pi < x < \pi)$$

$$f(x) = x^2 \quad (0 < x < 2\pi)$$

$$f(x) = \begin{cases} x^2 & \text{if } -\frac{1}{2}\pi < x < \frac{1}{2}\pi \\ \frac{1}{4}\pi^2 & \text{if } \frac{1}{2}\pi < x < \frac{3}{2}\pi \end{cases} \qquad f(x) = \begin{cases} -4x & \text{if } -\pi < x < 0 \\ 4x & \text{if } 0 < x < \pi \end{cases}$$

Even and Odd Functions :

Half-Range Expansions :

Theorem 1

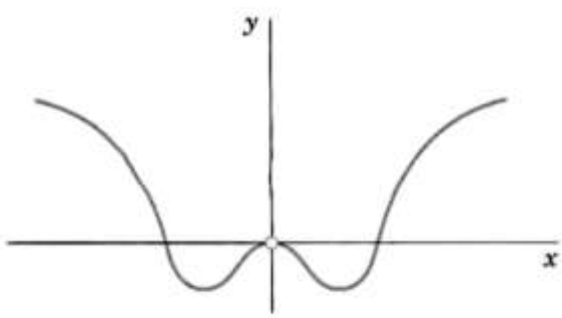


Fig. 262. Even function

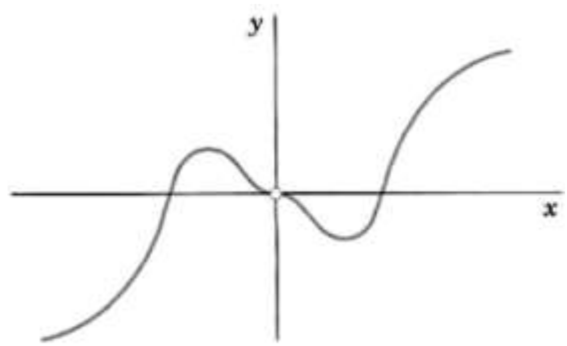


Fig. 263. Odd function

Fourier Cosine Series, Fourier Sine Series

The Fourier series of an **even** *function of period 2L is a* **"Fourier cosine series"**

(1)
$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x \qquad (f \text{ even})$$

with coefficients (note: integration from 0 to L only!)

(2)
$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx, \qquad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx, \qquad n = 1, 2, \cdots.$$

The Fourier series of an **odd** *function of period 2L is a* **"Fourier sine series"**

(3)
$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x \qquad (f \text{ odd})$$

with coefficients

(4)
$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.$$

Since the definite integral of a function gives the area under the curve of the function between the limits of integration, we have

$$\int_{-L}^{L} g(x)\,dx = 2\int_0^L g(x)\,dx \qquad \text{for even } g$$

$$\int_{-L}^{L} h(x)\,dx = 0 \qquad \text{for odd } h$$

The Case of Period 2π. If $L = \pi$, then $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ (f even) with coefficients

(2*) $$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx, \qquad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx, \qquad n = 1, 2, \cdots$$

and $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$ (f odd) with coefficients

(4*) $$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx, \qquad n = 1, 2, \cdots.$$

THEOREM 2

Sum and Scalar Multiple

The Fourier coefficients of a sum $f_1 + f_2$ are the sums of the corresponding Fourier coefficients of f_1 and f_2.

The Fourier coefficients of cf are c times the corresponding Fourier coefficients of f.

Example 1

Rectangular Pulse

The function $f^*(x)$ in Fig. 264 is the sum of the function $f(x)$ in Example 1 of Sec 11.1 and the constant k. Hence, from that example and Theorem 2 we conclude that

$$f^*(x) = k + \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right). \qquad \blacksquare$$

Example 2

Half-Wave Rectifier

The function $u(t)$ in Example 3 of Sec. 11.2 has a Fourier cosine series plus a single term $v(t) = (E/2)\sin \omega t$. We conclude from this and Theorem 2 that $u(t) - v(t)$ must be an even function. Verify this graphically. (See Fig. 265.) $\blacksquare$

Fig. 264. Example 1

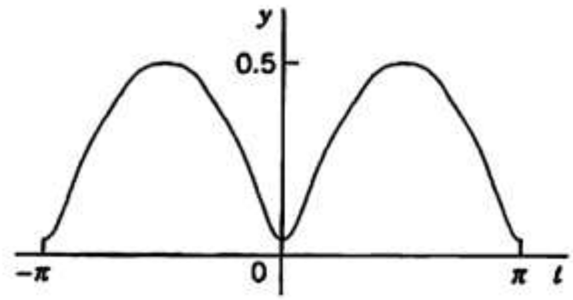


Fig. 265. $u(t) - v(t)$ with $E = 1$, $\omega = 1$

Sawtooth Wave

Find the Fourier series of the function (Fig. 266)

$$f(x) = x + \pi \quad \text{if} \quad -\pi < x < \pi \qquad \text{and} \qquad f(x + 2\pi) = f(x).$$

Solution. We have $f = f_1 + f_2$, where $f_1 = x$ and $f_2 = \pi$. The Fourier coefficients of f_2 are zero, except for the first one (the constant term), which is π. Hence, by Theorem 2, the Fourier coefficients a_n, b_n are those of f_1, except for a_0, which is π. Since f_1 is odd, $a_n = 0$ for $n = 1, 2, \cdots$, and

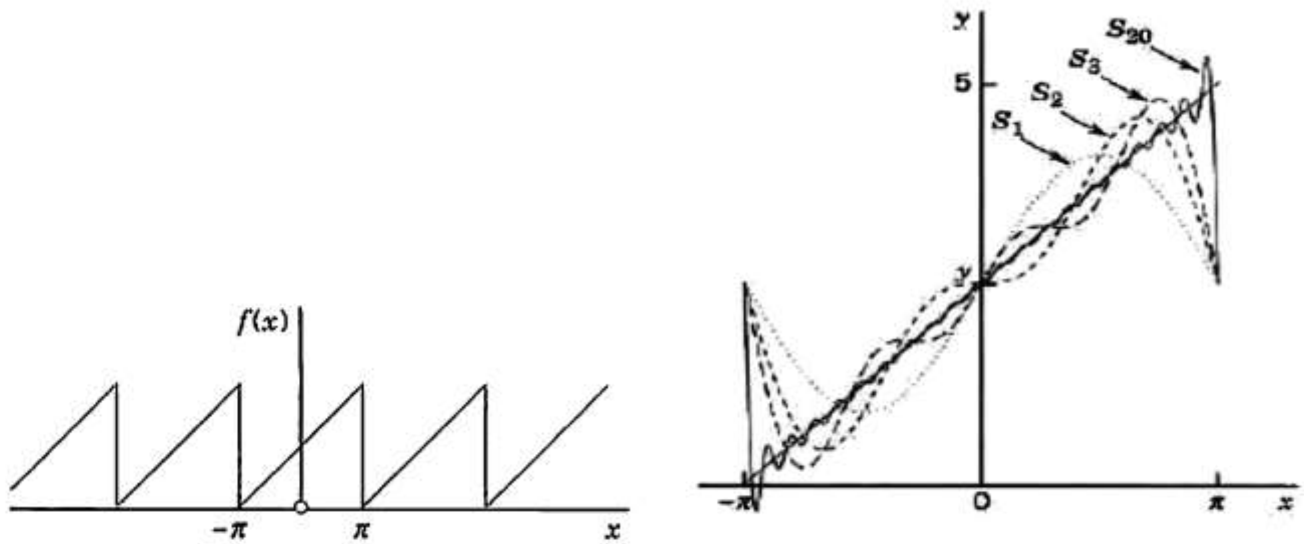
$$b_n = \frac{2}{\pi} \int_0^{\pi} f_1(x) \sin nx \, dx = \frac{2}{\pi} \int_0^{\pi} x \sin nx \, dx.$$

Integrating by parts, we obtain

$$b_n = \frac{2}{\pi} \left[\frac{-x \cos nx}{n} \bigg|_0^{\pi} + \frac{1}{n} \int_0^{\pi} \cos nx \, dx \right] = -\frac{2}{n} \cos n\pi.$$

Hence $b_1 = 2$, $b_2 = -2/2$, $b_3 = 2/3$, $b_4 = -2/4, \cdots$, and the Fourier series of $f(x)$ is

$$f(x) = \pi + 2 \left(\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - + \cdots \right).$$ ■



(*a*) The function $f(x)$

(*b*) Partial sums S_1, S_2, S_3, S_{20}

Fig. 266. Example 3

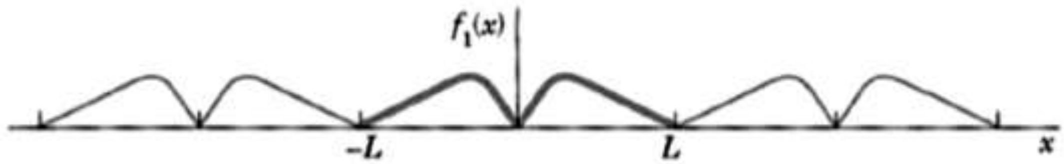
<u>Half-Range Expansions:</u>

Half range expansions are Fourier series. We want to represent f(x) in Fig 267a by a Fourier series.

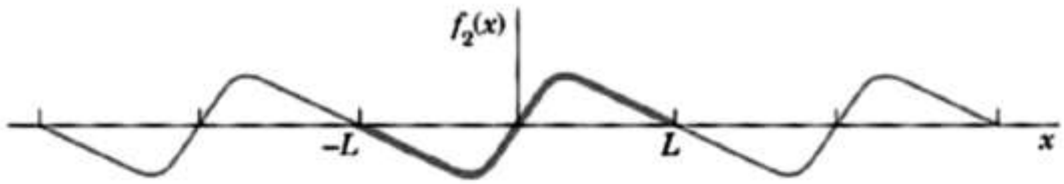
We could extend $f(x)$ as a function of period L and develop the extended function into a Fourier series. But this series would in general contain *both* cosine *and* sine terms. We can do better and get simpler series. Indeed, for our given f we can calculate Fourier coefficients from (2) or from (4) in Theorem 1. And we have a choice and can take what seems more practical. If we use (2), we get (1). This is the **even periodic extension** f_1 of f in Fig. 267b. If we choose (4) instead, we get (3), the **odd periodic extension** f_2 of f in Fig. 267c.

Both extensions have period $2L$. This motivates the name **half-range expansions:** f is given (and of physical interest) only on half the range, half the interval of periodicity of length $2L$.

(a) The given function $f(x)$



(b) $f(x)$ extended as an ***even*** periodic function of period $2L$



(c) $f(x)$ extended as an ***odd*** periodic function of period $2L$

Fig. 267. (a) Function $f(x)$ given on an interval $0 \leqq x \leqq L$

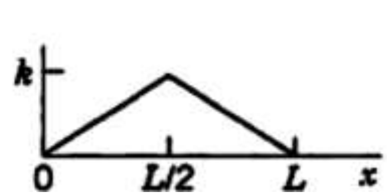
(b) Even extension to the full "range" (interval) $-L \leqq x \leqq L$ (heavy curve) and the periodic extension of period $2L$ to the x-axis

(c) Odd extension to $-L \leqq x \leqq L$ (heavy curve) and the periodic extension of period $2L$ to the x-axis

EXAMPLE 4 "Triangle" and Its Half-Range Expansions

Find the two half-range expansions of the function (Fig. 268)

$$f(x) = \begin{cases} \dfrac{2k}{L}x & \text{if } 0 < x < \dfrac{L}{2} \\ \dfrac{2k}{L}(L - x) & \text{if } \dfrac{L}{2} < x < L. \end{cases}$$



Solution. ***(a) Even periodic extension.*** From (2) we obtain

$$a_0 = \frac{1}{L}\left[\frac{2k}{L}\int_0^{L/2} x\,dx + \frac{2k}{L}\int_{L/2}^{L}(L-x)\,dx\right] = \frac{k}{2}.$$

$$a_n = \frac{2}{L}\left[\frac{2k}{L}\int_0^{L/2} x\cos\frac{n\pi}{L}x\,dx + \frac{2k}{L}\int_{L/2}^{L}(L-x)\cos\frac{n\pi}{L}x\,dx\right].$$

We consider a_n. For the first integral we obtain by integration by parts

$$\int_0^{L/2} x\cos\frac{n\pi}{L}x\,dx = \frac{Lx}{n\pi}\sin\frac{n\pi}{L}x\Big|_0^{L/2} - \frac{L}{n\pi}\int_0^{L/2}\sin\frac{n\pi}{L}x\,dx$$

$$= \frac{L^2}{2n\pi}\sin\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right).$$

Similarly, for the second integral we obtain

$$\int_{L/2}^{L}(L-x)\cos\frac{n\pi}{L}x\,dx = \frac{L}{n\pi}(L-x)\sin\frac{n\pi}{L}x\Big|_{L/2}^{L} + \frac{L}{n\pi}\int_{L/2}^{L}\sin\frac{n\pi}{L}x\,dx$$

$$= \left(0 - \frac{L}{n\pi}\left(L - \frac{L}{2}\right)\sin\frac{n\pi}{2}\right) - \frac{L^2}{n^2\pi^2}\left(\cos n\pi - \cos\frac{n\pi}{2}\right).$$

We insert these two results into the formula for a_n. The sine terms cancel and so does a factor L^2. This gives

$$a_n = \frac{4k}{n^2\pi^2}\left(2\cos\frac{n\pi}{2} - \cos n\pi - 1\right).$$

Thus,

$$a_2 = -16k/(2^2\pi^2), \qquad a_6 = -16k/(6^2\pi^2), \qquad a_{10} = -16k/(10^2\pi^2), \cdots$$

and $a_n = 0$ if $n \neq 2, 6, 10, 14, \cdots$. Hence the first half-range expansion of $f(x)$ is (Fig. 269a)

$$f(x) = \frac{k}{2} - \frac{16k}{\pi^2}\left(\frac{1}{2^2}\cos\frac{2\pi}{L}x + \frac{1}{6^2}\cos\frac{6\pi}{L}x + \cdots\right).$$

This Fourier cosine series represents the even periodic extension of the given function $f(x)$, of period $2L$.

(b) ***Odd periodic extension.*** Similarly, from (4) we obtain

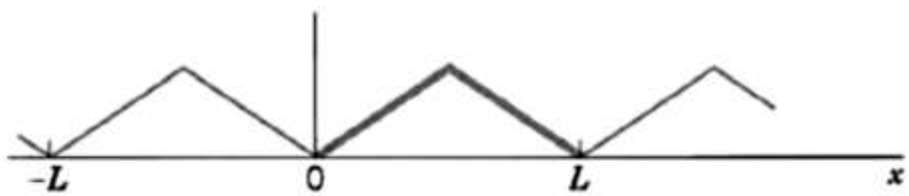
$$b_n = \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}. \tag{5}$$

Hence the other half-range expansion of $f(x)$ is (Fig. 269b)

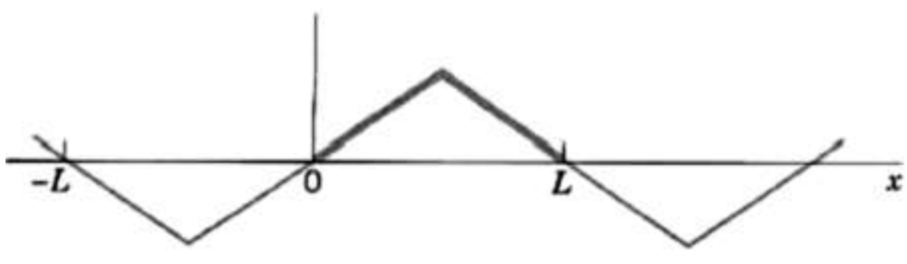
$$f(x) = \frac{8k}{\pi^2}\left(\frac{1}{1^2}\sin\frac{\pi}{L}x - \frac{1}{3^2}\sin\frac{3\pi}{L}x + \frac{1}{5^2}\sin\frac{5\pi}{L}x - + \cdots\right).$$

This series represents the odd periodic extension of $f(x)$, of period $2L$.

Basic applications of these results will be shown in Secs. 12.3 and 12.5.



(a) Even extension



(b) Odd extension

Fig. 269. Periodic extensions of $f(x)$ in Example 4

Fourier Series Summary and Notes:

1. Suppose $f(x)$ is a periodic function of period 2π which can be represented by a **TRIGONOMETRIC FOURIER SERIES**

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx.$$

(This means that the series above converges to $f(x)$.)

Then the **Fourier Coefficients** satisfy the **Euler Formulae**, namely:

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \quad \text{for } n = 1, 2, \ldots \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \quad \text{for } n = 1, 2, \ldots \end{aligned}$$

Recall the product of two even functions is even, the product of two odd functions is even and the product of an even and an odd function is odd. Compare

3. If f is an **odd** function then

$$\int_{-\pi}^{\pi} f(x)\,dx = 0,$$

while if f is an even function, then

$$\int_{-\pi}^{\pi} f(x)\,dx = 2\int_{0}^{\pi} f(x)\,dx$$

Review of integration by parts:

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

It is useful when f can be differentiated repeatedly and g can be integrated repeatedly without difficulty. The integral

$$\int xe^x\,dx$$

is such an integral because $f(x) = x$ can be differentiated twice to become zero and $g(x) = e^x$ can be integrated repeatedly without difficulty. Integration by parts also applies to integrals like

$$\int e^x \sin x\,dx$$

Sometimes it is easier to remember the formula if we write it in differential form. Let $u = f(x)$ and $v = g(x)$. Then $du = f'(x)\,dx$ and $dv = g'(x)\,dx$. Using the Substitution Rule, the integration by parts formula becomes

Integration by Parts Formula

$$\int u\,dv = uv - \int v\,du$$

EXAMPLE 1 Using Integration by Parts

Find

$$\int x\cos x\,dx.$$

Solution We use the formula $\int u\,dv = uv - \int v\,du$ with

$$u = x, \qquad dv = \cos x\,dx,$$
$$du = dx, \qquad v = \sin x.$$

Then

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C.$$

EXAMPLE 2

Integral of the Natural Logarithm

Find

$$\int \ln x\,dx.$$

Solution Since $\int \ln x\,dx$ can be written as $\int \ln x \cdot 1\,dx$, we use the formula $\int u\,dv = uv - \int v\,du$ with

$u = \ln x$ Simplifies when differentiated $\qquad dv = dx$ Easy to integrate

$du = \frac{1}{x}\,dx,$ $\qquad v = x.$ Simplest antiderivative

Then

$$\int \ln x\,dx = x\ln x - \int x \cdot \frac{1}{x}\,dx = x\ln x - \int dx = x\ln x - x + C.$$

EXAMPLE 3:

Evaluate

$$\int x^2 e^x \, dx.$$

Solution With $u = x^2$, $dv = e^x\,dx$, $du = 2x\,dx$, and $v = e^x$, we have

$$\int x^2 e^x \, dx = x^2 e^x - 2\int x e^x \, dx.$$

The new integral is less complicated than the original because the exponent on x is reduced by one. To evaluate the integral on the right, we integrate by parts again with $u = x$, $dv = e^x\,dx$. Then $du = dx$, $v = e^x$, and

$$\int x e^x \, dx = x e^x - \int e^x \, dx = x e^x - e^x + C.$$

Thus;

$$\int x^2 e^x \, dx = x^2 e^x - 2\int x e^x \, dx$$

$$= x^2 e^x - 2x e^x + 2e^x + C.$$

EXAMPLE 4:

Evaluate

$$\int e^x \cos x \, dx.$$

Solution Let $u = e^x$ and $dv = \cos x\,dx$. Then $du = e^x\,dx$, $v = \sin x$, and

$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx.$$

The second integral is like the first except that it has $\sin x$ in place of $\cos x$. To evaluate it, we use integration by parts with

$$u = e^x, \qquad dv = \sin x\,dx, \qquad v = -\cos x, \qquad du = e^x\,dx.$$

Then;

$$\int e^x \cos x\,dx = e^x \sin x - \left(-e^x \cos x - \int (-\cos x)(e^x\,dx)\right)$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x\,dx.$$

The unknown integral now appears on both sides of the equation. Adding the integral to both sides and adding the constant of integration gives

$$2\int e^x \cos x\,dx = e^x \sin x + e^x \cos x + C_1.$$

Dividing by 2 and renaming the constant of integration gives

$$\int e^x \cos x\,dx = \frac{e^x \sin x + e^x \cos x}{2} + C.$$

Exercises and Examples

Example 1. *Let f be a periodic function of period 2π such that*

$$f(x) = \pi^2 - x^2 \quad \text{for } x \in (-\pi, \pi).$$

Find the Fourier series expansion.

- Check whether f is even or odd.
- If f is **odd**, all the Fourier coefficients $\mathbf{a_n}$ for $n = 0, 1, 2\ldots$ are **zero**; if f is **even**, all the Fourier coefficients $\mathbf{b_n}$ for $n = 1, 2\ldots$ are **zero**.
- Compute the remaining Fourier coefficients using the Euler Formulae. It is generally a good strategy to use **Integration by Parts**, successively **integrating** $\sin nx$ **and** $\cos nx$ and **differentiating** $f(x)$.
- Replace the expressions for the Fourier coefficients a_n, b_n in

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx.$$

STEP 1: $f(-x) = \pi^2 - (-x)^2 = \pi^2 - x^2 = f(x)$ so f is even.

STEP 2: Since $f(x)$ is even and $\sin nx$ is odd, $f(x) \sin nx$ is odd and hence

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = 0.$$

STEP 3: Since $f(x)$ is even and $\cos nx$ is even, $f(x) \cos nx$ is even, and so

$$\int_{-\pi}^{\pi} f(x) \cos nx \, dx = 2 \int_{0}^{\pi} f(x) \cos nx \, dx.$$

Therefore,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}2\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi}(\pi^2 - x^2)\cos nx$$

---------------------------- eq (a)

Now; we use the **Integration by parts method**

$$\int_0^{\pi} \underbrace{(\pi^2 - x^2)}_{f}\underbrace{\cos nx}_{g'}\,dx$$

$$= \underbrace{(\pi^2 - x^2)}_{f}\underbrace{\frac{1}{n}\sin nx}_{g}\Big|_0^{\pi} - \int_0^{\pi}\underbrace{-2x}_{f'}\underbrace{\frac{1}{n}\sin nx}_{g}\,dx$$

$$= (\pi^2 - \pi^2)\frac{1}{n}\sin n\pi - (\pi^2 - 0^2)\frac{1}{n}\sin 0 + \frac{2}{n}\int_0^{\pi} x\sin nx\,dx$$

$$= \frac{2}{n}\int_0^{\pi} x\sin nx\,dx.$$

---------------------------- eq (b)

Using Integration By Parts again . . . .

$$\int_0^{\pi}\underbrace{x}_{f}\underbrace{\sin nx}_{g'}\,dx = \underbrace{x}_{f}\underbrace{\frac{-\cos nx}{n}}_{g}\Big|_0^{\pi} - \int_0^{\pi}\underbrace{1}_{f'}\underbrace{\frac{-\cos nx}{n}}_{g}\,dx$$

$$= \pi\frac{-\cos n\pi}{n} - 0\frac{-\cos 0}{n} + \frac{1}{n}\int_0^{\pi}\cos nx\,dx$$

$$= -\frac{1}{n}\pi(-1)^n + \frac{1}{n}\frac{\sin nx}{n}\Big|_0^{\pi} = -\frac{1}{n}\pi(-1)^n.$$

Substituting in equation (b0 and (a) to get ;

$$a_n = \frac{2}{\pi}\int_0^{\pi}(\pi^2 - x^2)\cos nx\,dx = \frac{2}{\pi}\frac{2}{n}\int_0^{\pi} x\cos nx\,dx$$
$$= \frac{2}{\pi}\frac{2}{n} - \frac{1}{n}\pi(-1)^n = \frac{-4}{n^2}(-1)^n.$$

It remains to calculate a_0, which is given by

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}2\int_0^{\pi}\pi^2 - x^2\,dx \\ &= \frac{1}{\pi}\left(\pi^2 x - \frac{x^3}{3}\right)\Big|_0^{\pi} = \frac{1}{\pi}\left(\pi^3 - \frac{\pi^3}{3}\right) = \frac{2\pi^3}{3} \end{aligned}$$

where we use the fact that $f(x) = \pi^2 - x^2$ is even.

$$\pi^2 - x^2 = \frac{2\pi^3}{3} + \sum_{n=1}^{\infty}\frac{-4}{n^2}(-1)^n\cos nx + 0.\sin nx = \frac{2\pi^3}{3} + \sum_{n=1}^{\infty}\frac{-4}{n^2}(-1)^n\cos nx$$

Example 2. *Show that the trigonometric Fourier series of $f(x) = 3x$ for $x \in (-\pi, \pi)$ is given by*

$$\sum_{n=1}^{\infty}\frac{-6}{n}(-1)^n\sin nx.$$

SOLUTION:

STEP 1: $f(-x) = 3.-x = -3x = -f(x)$, so f is an odd function.

STEP 2: Since $f(x)$ is odd and $\cos nx$ is even, it follows that $f(x)\cos nx$ is odd, so

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}.0 = 0.$$

Moreover, since f is odd

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}.0 = 0.$$

STEP 3: We need to calculate the Fourier coefficients using the Euler Formulae. However, noting that $f(x)$ and $\sin nx$ are odd, and therefore that $f(x)\sin nx$ is even we have

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}2\int_{0}^{\pi} f(x)\sin nx\,dx = \frac{6}{\pi}\int_{0}^{\pi} x\sin nx\,dx$$

------------------------------ eq (c)

The latter integral is calculated using integration by parts.

$$\int_0^{\pi} x\sin nx\,dx = x\frac{-\cos nx}{n}\Big|_0^{\pi} - \int_0^{\pi}\frac{-\cos nx}{n}\,dx = \frac{-\pi}{n}(-1)^n.$$

Substituting in to equation (c), get;

$$b_n = \frac{6}{\pi}\frac{-\pi}{n}(-1)^n = -\frac{6}{n}(-1)^n.$$

STEP 4: The Fourier series of $f(x) = 3x$ is given by

$$\begin{aligned} a_0 + \sum_{n=1}^{\infty} a_n\cos nx + b_n\sin nx &= 0 + \sum_{n=1}^{\infty} 0\cos nx + -\frac{6}{n}(-1)^n\sin nx \\ &= \sum_{n=1}^{\infty} -\frac{6}{n}(-1)^n\sin nx. \end{aligned}$$

Home Work1:

Show that the Fourier Series expansion of x^3 is:

$$x^3 = \sum_{n=1}^{\infty} 2(-1)^n \left(\frac{6}{n^3} - \frac{\pi^2}{n}\right) \sin nx.$$

Home Work2:

Let $f(x)$ be defined by

$$f(x) = \begin{cases} 0, & -3 < x < -1, \\ 1, & -1 < x < 1, \\ 0, & 1 < x < 3 \end{cases}$$

$$f(x+6) = f(x) \quad \text{for all } x.$$

Find the Fourier series for f.

ANSWER:

$$a_0 = \frac{2}{3}, \quad a_n = \frac{2}{n\pi} sin(\frac{n\pi}{3}), \quad b_n = 0$$

Example 3 : Find the F.S. expansion for ;

$f(x) = |x|$, on $-\pi < x < \pi$

Solution: |X| is an even function**, so**

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\, dx = \frac{2}{\pi}\int_0^{\pi} x\, dx = \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dx$$

$$= \frac{2}{\pi}\left(\left[\frac{x\sin(nx)}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{\sin(nx)}{n}\,dx\right) = \frac{2}{\pi}\left[\frac{\cos(nx)}{n^2}\right]_0^{\pi}$$

$$= \frac{2(-1)^n}{\pi n^2} - \frac{2}{\pi n^2} = \frac{(2(-1)^n - 1)}{\pi n^2}$$

Since the function is even and the sine is odd, then b_n=0. It will be shown in details below;

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\sin(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -x\sin(nx)\,dx + \frac{1}{\pi}\int_0^{\pi} x\sin(nx)\,dx$$

$$= \frac{1}{\pi}\left[\left[\frac{x\cos(nx)}{n}\right]_{-\pi}^{0} - \int_{-\pi}^{0}\frac{\cos(nx)}{n}\,dx\right] + \frac{1}{n}\left[\left[\frac{-x\cos(nx)}{n}\right]_0^{\pi} + \int_0^{\pi}\frac{\cos(nx)}{n}\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi(-1)^n}{n} - \left[\frac{\sin(nx)}{n^2}\right]_{-\pi}^{0}\right] + \frac{1}{\pi}\left[\frac{-\pi(-1)^n}{n} + \left[\frac{\sin(nx)}{n^2}\right]_0^{\pi}\right]$$

$$= \frac{1}{\pi}\frac{\pi(-1)^n}{n} + \frac{1}{\pi}\frac{-\pi(-1)^n}{n} = 0$$

So on the interval $[-\pi, \pi]$, our function is:

$$|x| = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\cos(nx)$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{1}{9}\cos(3x) + \frac{1}{25}\cos(5x) + \ldots\right)$$

Home Work3:

Find the Fourier Series Expansion of the function:

$f(x) = 1 + x$ on $[-\pi, \pi]$ **Answer:** $$\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nx)$$

Example 4:

Find the F.S. expansion for ;

$$f(x) = \begin{cases} 1 & -1 \le x < 0 \\ \frac{1}{2} & x = 0 \\ x & 0 < x \le 1 \end{cases} \quad \text{on } [-1, 1]$$

$$a_n = \int_{-1}^{1} f(x) \cos(n\pi x)\, dx$$

$$= \int_{-1}^{0} f(x) \cos(n\pi x)\, dx + \int_{0}^{1} f(x) \cos(n\pi x)\, dx$$

$$= \int_{-1}^{0} \cos(n\pi x)\, dx + \int_{0}^{1} x \cos(n\pi x)\, dx$$

$$a_n = \left[\frac{\sin(n\pi x)}{n\pi}\right]_{-1}^{0} + \left[\frac{x \sin(n\pi x)}{n\pi} + \frac{\cos(n\pi x)}{n^2\pi^2}\right]_{0}^{1}$$

$$= \frac{\cos(n\pi) - 1}{n^2\pi^2} = \frac{(-1)^n - 1}{n^2\pi^2} \quad n = 1, 2, 3, \ldots$$

$$a_0 = \int_{-1}^{1} f(x)\, dx = \int_{-1}^{0} dx + \int_{0}^{1} x dx = \frac{3}{2}$$

$$b_n = \int_{-1}^{1} f(x)\sin(n\pi x)\,dx$$

$$= \int_{-1}^{0} f(x)\sin(n\pi x)\,dx + \int_{0}^{1} f(x)\sin(n\pi x)\,dx$$

$$= \int_{-1}^{0} \sin(n\pi x)\,dx + \int_{0}^{1} x\,\sin(n\pi x)\,dx$$

$$= \left[-\frac{\cos(n\pi x)}{n\pi}\right]_{-1}^{0} + \left[-\frac{x\,\cos(n\pi x)}{n\pi} + \frac{\sin(n\pi x)}{n^2\pi^2}\right]_{0}^{1} = -\frac{1}{n\pi}$$

So the Fourier series is:

$$f(x) = \frac{3}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2\pi^2}\cos(n\pi x) - \frac{1}{n\pi}\sin(n\pi x)$$

Home Work4:

Find the Fourier Series Expansion of the function:

$f(x) = x^2$ on $[-\pi, \pi]$ **Answer** $x^2 = \frac{\pi^2}{3} + \sum_{k=1}^{\infty} \frac{4(-1)^k}{k^2}\cos(kx)$

Exercises:

1- Find the Fourier series expansion for the periodic function f(t) in Fig.1.

2- Find the half range Fourier series expansion for the non-periodic function m(t) in Fig.2.

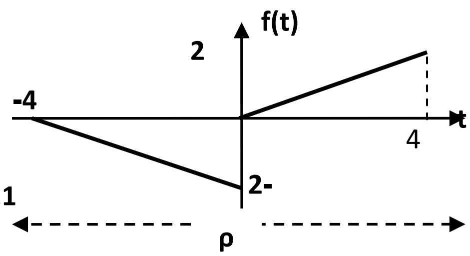


Fig.1

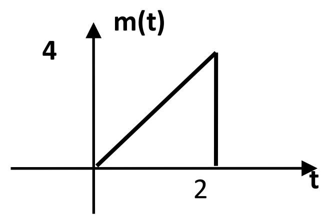


Fig.2

3- Find the Fourier series expansion for the periodic function f(t) in Fig.3.

4- Find the half range Fourier series expansion for the non-periodic function m(t) in Fig.4.

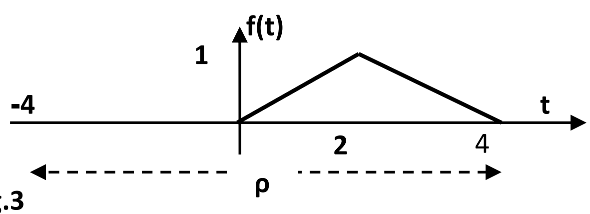


Fig.3

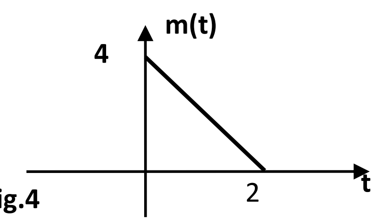


Fig.4

5- Find the Fourier series expansion for the periodic function f(t) in Fig.5.

6- Find the half range Fourier series expansion for the non-periodic function m(t) in Fig.6.

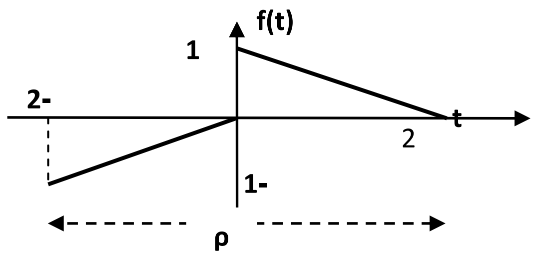


Fig.5

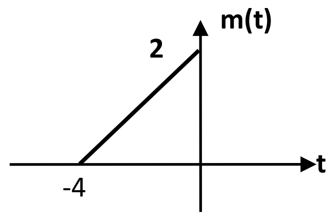


Fig.6

Fourier Transform

Let f(t) be the time domain non-periodic function. Then the frequency domain Fourier transform **FT** (**exponential form**) corresponding function is

$$\boldsymbol{F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t} \; dt}$$

While the Inverse Fourier Transform (IFT) is defined as;

$$\boldsymbol{f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t} \; d\omega}$$

Example:

Find the FT of f(t)=1 $|t| > 1$

Solution:

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} 1 \,.\, e^{-j\omega t} \; dt = \frac{1}{-j\omega\sqrt{2\pi}} \left[e^{-j\omega t}\right]_{-1}^{1} = \frac{1}{-j\omega\sqrt{2\pi}} \left(e^{-j\omega} - e^{j\omega}\right)$$

$$F(\omega) = \frac{2}{\omega\sqrt{2\pi}} \sin\omega = \sqrt{\frac{2}{\pi}} \; \frac{\sin\omega}{\omega}$$

Example:

Find the FT of $f(t) = \begin{cases} e^{-at} & for \;\; t \geq 0 \\ 0 & for \; t < 0 \end{cases}$ where a is a constant

Solution:

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-at} \,.\, e^{-j\omega t}\, dt = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt =$$

$$F(\omega) = \frac{-1}{\sqrt{2\pi}\ (a+jw)} \left[e^{-(a+j\omega)t}\right]_0^\infty = \frac{\mathbf{1}}{\sqrt{\mathbf{2\pi}}\ \ (\boldsymbol{a}+\boldsymbol{jw})}$$

Exercises:

1- $f(x) = \begin{cases} e^{kx} & \text{if } x < 0 \quad (k > 0) \\ 0 & \text{if } x > 0 \end{cases}$

2- $f(x) = \begin{cases} k & \text{if } 0 < x < b \\ 0 & \text{otherwise} \end{cases}$

3- $f(x) = \begin{cases} e^{2ix} & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

4- $f(x) = \begin{cases} xe^{-x} & \text{if } -1 < x < 0 \\ 0 & \text{otherwise} \end{cases}$

5- $f(x) = \begin{cases} k & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

6- $f(x) = \begin{cases} x & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

7- $f(x) = \begin{cases} x & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

8- $f(x) = \begin{cases} -1 & \text{if } -1 < x < 0 \\ 1 & \text{if } \ 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

9- $f(t) = \sin 2\pi\, t \qquad 0 \le t \le \frac{\pi}{2}$

10- $f(t) = \cos 2\pi\, t \qquad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$

Some Useful Identities

$$\sin(-\theta) = -\sin\theta, \quad \cos(-\theta) = \cos\theta$$

$$\sin^2\theta + \cos^2\theta = 1, \quad \sec^2\theta = 1 + \tan^2\theta, \quad \csc^2\theta = 1 + \cot^2\theta$$

$$\sin 2\theta = 2\sin\theta\cos\theta, \quad \cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2}, \quad \sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\sin(A - B) = \sin A\cos B - \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\cos(A - B) = \cos A\cos B + \sin A\sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}, \qquad \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

$$\sin\left(A - \frac{\pi}{2}\right) = -\cos A, \qquad \cos\left(A - \frac{\pi}{2}\right) = \sin A$$

$$\sin\left(A + \frac{\pi}{2}\right) = \cos A, \qquad \cos\left(A + \frac{\pi}{2}\right) = -\sin A$$

$$\sin A\sin B = \frac{1}{2}\cos(A - B) - \frac{1}{2}\cos(A + B)$$

$$\cos A\cos B = \frac{1}{2}\cos(A - B) + \frac{1}{2}\cos(A + B)$$

$$\sin A\cos B = \frac{1}{2}\sin(A - B) + \frac{1}{2}\sin(A + B)$$

$$\sin A + \sin B = 2\sin\frac{1}{2}(A + B)\cos\frac{1}{2}(A - B)$$

$$\sin A - \sin B = 2\cos\frac{1}{2}(A + B)\sin\frac{1}{2}(A - B)$$

$$\cos A + \cos B = 2\cos\frac{1}{2}(A + B)\cos\frac{1}{2}(A - B)$$

$$\cos A - \cos B = -2\sin\frac{1}{2}(A + B)\sin\frac{1}{2}(A - B)$$

Some Useful Integrations

$$\int uv'\,dx = uv - \int u'v\,dx$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c \qquad (n \neq -1$$

$$\int \frac{1}{x}\,dx = \ln|x| + c$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$$

$$\int \sin x\,dx = -\cos x + c$$

$$\int \cos x\,dx = \sin x + c$$

$$\int \tan x\,dx = -\ln|\cos x| + c$$

$$\int \cot x\,dx = \ln|\sin x| + c$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + c$$

$$\int \csc x\,dx = \ln|\csc x - \cot x| + c$$

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\arctan\frac{x}{a} + c$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\frac{x}{a} + c$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \text{arcsinh}\,\frac{x}{a} + c$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \text{arccosh}\,\frac{x}{a} + c$$

$$\int \sin^2 x\,dx = \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x +$$

$$\int \cos^2 x\,dx = \tfrac{1}{2}x + \tfrac{1}{4}\sin 2x +$$

$$\int \tan^2 x\,dx = \tan x - x + c$$

$$\int \cot^2 x\,dx = -\cot x - x + c$$

$$\int \ln x\,dx = x\ln x - x + c$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2 + b^2}(a\sin bx - b\cos bx) + c$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2 + b^2}(a\cos bx + b\sin bx) + c$$