



## VECTOR INTEGRAL CALCULUS

### Line Integral (Curve Integral):

We represent the curve  $C$  by a parametric representation

$$\mathbf{r}(t) = [x(t), y(t), z(t)] = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \quad (a \leq t \leq b).$$

The curve  $C$  is called the **path of integration**,  $A: \mathbf{r}(a)$  its *initial point*, and  $B: \mathbf{r}(b)$  its terminal point.  $C$  is now *oriented*. The direction from  $A$  to  $B$ , in which  $t$  increases, is called the *positive direction* on  $C$  and can be marked by an arrow as in fig. 1

**A line integral** of a vector function  $\mathbf{F}(\mathbf{r})$  over a curve  $C: \mathbf{r}(t)$  is defined by

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt$$

$\uparrow$   
 Dot product

$$\mathbf{r}' = \frac{d\mathbf{r}}{dt}$$

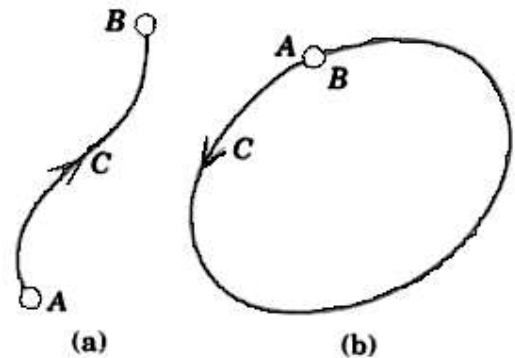


Fig. 1 Oriented curve

Or ;

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_C (F_1 dx + F_2 dy + F_3 dz)$$

$$= \int_a^b (F_1 x' + F_2 y' + F_3 z') dt.$$

$\int_C$  also written as  $\oint_C$ .

### Example 1 :

Find the value of the line integral when  $\mathbf{F}(\mathbf{r}) = -y\mathbf{i} - xy\mathbf{j}$  and  $C$  is the circular arc in Fig. 2 from  $A$  to  $B$ .

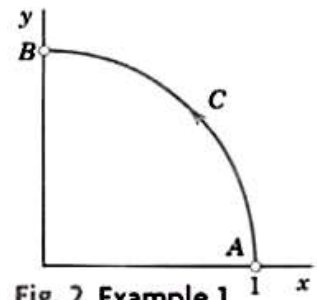


Fig. 2 Example 1



### Solution:

We may represent  $C$  by  $\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j}$ , where  $0 \leq t \leq \pi/2$ . Then  
 $x(t) = \cos t$ ,  $y(t) = \sin t$ , and

$$\mathbf{F}(\mathbf{r}(t)) = -y(t)\mathbf{i} - x(t)y(t)\mathbf{j} = -\sin t \mathbf{i} - \cos t \sin t \mathbf{j}.$$

By differentiation,  $\mathbf{r}'(t) = \frac{d}{dt} (\cos t \mathbf{i} + \sin t \mathbf{j}) = -\sin t \mathbf{i} + \cos t \mathbf{j}$

$$\begin{aligned} \int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} &= \int_0^{\pi/2} (-\sin t \mathbf{i} - \cos t \sin t \mathbf{j}) \cdot (-\sin t \mathbf{i} + \cos t \mathbf{j}) dt = \int_0^{\pi/2} (\sin^2 t - \cos^2 t \sin t) dt \\ &= \int_0^{\pi/2} \frac{1}{2} (1 - \cos 2t) dt - \int_1^0 u^2 (-du) = \frac{\pi}{4} - 0 - \frac{1}{3} \approx 0.4521. \quad \blacksquare \end{aligned}$$

### Line Integral in space (3 Dimensional):

#### Example 2:

The evaluation of line integrals in space is practically the same as it is in the plane. To see this, find the value of  $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$  when  $\mathbf{F}(\mathbf{r}) = [z, x, y] = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$  and  $C$  is the helix (Fig. 3 )

$$\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + 3t\mathbf{k} \quad (0 \leq t \leq 2\pi).$$

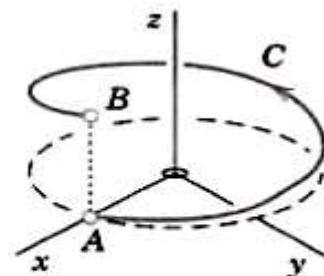


Fig. 3 Example 2

### Solution:

we have  $x(t) = \cos t$ ,  $y(t) = \sin t$ ,  $z(t) = 3t$ . Thus

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = (3t\mathbf{i} + \cos t \mathbf{j} + \sin t \mathbf{k}) \cdot (-\sin t \mathbf{i} + \cos t \mathbf{j} + 3\mathbf{k}).$$

The dot product is  $3t(-\sin t) + \cos^2 t + 3 \sin t$ . Hence

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_0^{2\pi} (-3t \sin t + \cos^2 t + 3 \sin t) dt = 6\pi + \pi + 0 = 7\pi \approx 21.99. \quad \blacksquare$$



### Example 3:

Evaluate  $\int_{(0,1)}^{(1,2)} (x^2 - y) dx + (y^2 + x) dy$  along (a) a straight line from (0, 1) to (1, 2), (b) straight lines from (0, 1) to (1, 1) and then from (1, 1) to (1, 2), (c) the parabola  $x = t, y = t^2 + 1$ .

### Solution:

(a) An equation for the line joining (0, 1) and (1, 2) in the  $xy$  plane is  $y = x + 1$ . Then  $dy = dx$  and the line integral equals

$$\int_{x=0}^1 \{x^2 - (x+1)\} dx + \{(x+1)^2 + x\} dx = \int_0^1 (2x^2 + 2x) dx = 5/3$$

(b) Along the straight line from (0, 1) to (1, 1),  $y = 1, dy = 0$  and the line integral equals

$$\int_{x=0}^1 (x^2 - 1) dx + (1+x)(0) = \int_0^1 (x^2 - 1) dx = -2/3$$

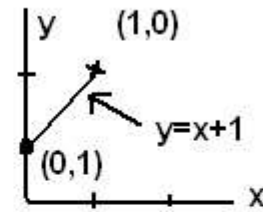
Along the straight line from (1, 1) to (1, 2),  $x = 1, dx = 0$  and the line integral equals

$$\int_{y=1}^2 (1-y)(0) + (y^2 + 1) dy = \int_1^2 (y^2 + 1) dy = 10/3$$

Then the required value =  $-2/3 + 10/3 = 8/3$ .

(c) Since  $t = 0$  at (0, 1) and  $t = 1$  at (1, 2), the line integral equals

$$\int_{t=0}^1 \{t^2 - (t^2 + 1)\} dt + \{(t^2 + 1)^2 + t\} 2t dt = \int_0^1 (2t^5 + 4t^3 + 2t^2 + 2t - 1) dt = 2$$



### Example 4:

If  $\mathbf{A} = (3x^2 - 6yz)\mathbf{i} + (2y + 3xz)\mathbf{j} + (1 - 4xyz^2)\mathbf{k}$ , evaluate  $\int_C \mathbf{A} \cdot d\mathbf{r}$  from (0, 0, 0) to (1, 1, 1) along the following paths C:

(a)  $x = t, y = t^2, z = t^3$

(b) The straight lines from (0, 0, 0) to (0, 0, 1), then to (0, 1, 1), and then to (1, 1, 1)

(c) The straight line joining (0, 0, 0) and (1, 1, 1)

$$\begin{aligned} \int_C \mathbf{A} \cdot d\mathbf{r} &= \int_C \{(3x^2 - 6yz)\mathbf{i} + (2y + 3xz)\mathbf{j} + (1 - 4xyz^2)\mathbf{k}\} \cdot (dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}) \\ &= \int_C (3x^2 - 6yz) dx + (2y + 3xz) dy + (1 - 4xyz^2) dz \end{aligned}$$



### Solution:

(a) If  $x = t, y = t^2, z = t^3$ , points  $(0, 0, 0)$  and  $(1, 1, 1)$  correspond to  $t = 0$  and  $t = 1$ , respectively. Then

$$\begin{aligned}\int_C \mathbf{A} \cdot d\mathbf{r} &= \int_{t=0}^1 \{3t^2 - 6(t^2)(t^3)\} dt + \{2t^2 + 3(t)(t^3)\} d(t^2) + \{1 - 4(t)(t^2)(t^3)^2\} d(t^3) \\ &= \int_{t=0}^1 (3t^2 - 6t^5) dt + (4t^3 + 6t^5) dt + (3t^2 - 12t^{11}) dt = 2\end{aligned}$$

**Another method:**

Along  $C$ ,  $\mathbf{A} = (3t^2 - 6t^5)\mathbf{i} + (2t^2 + 3t^4)\mathbf{j} + (1 - 4t^9)\mathbf{k}$  and  $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ ,  $d\mathbf{r} = (\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}) dt$ . Then

$$\int_C \mathbf{A} \cdot d\mathbf{r} = \int_0^1 (3t^2 - 6t^5) dt + (4t^3 + 6t^5) dt + (3t^2 - 12t^{11}) dt = 2$$

(b)

Along the straight line from  $(0, 0, 0)$  to  $(0, 1, 1)$ ,  $x = 0, y = 0, dx = 0, dy = 0$ , while  $z$  varies from 0 to 1. Then the integral over this part of the path is

$$\int_{z=0}^1 \{3(0)^2 - 6(0)(z)\}0 + \{2(0) + 3(0)(z)\}0 + \{1 - 4(0)(0)(z^2)\} dz = \int_{z=0}^1 dz = 1$$

Along the straight line from  $(0, 0, 1)$  to  $(0, 1, 1)$ ,  $x = 0, z = 1, dx = 0, dz = 0$ , while  $y$  varies from 0 to 1. Then the integral over this part of the path is

$$\int_{y=0}^1 \{3(0)^2 - 6(y)(1)\}0 + \{2y + 3(0)(1)\} dy + \{1 - 4(0)(y)(1)^2\}0 = \int_{y=0}^1 2y dy = 1$$

Along the straight line from  $(0, 1, 1)$  to  $(1, 1, 1)$ ,  $y = 1, z = 1, dy = 0, dz = 0$ , while  $x$  varies from 0 to 1. Then the integral over this part of the path is

$$\int_{x=0}^1 \{3x^2 - 6(1)(1)\} dx + \{2(1) + 3x(1)\}0 + \{1 - 4x(1)(1)^2\}0 = \int_{x=0}^1 (3x^2 - 6) dx = -5$$

Adding,  $\int_C \mathbf{A} \cdot d\mathbf{r} = 1 + 1 - 5 = -3.$

(c) The straight line joining  $(0, 0, 0)$  and  $(1, 1, 1)$  is given in parametric form by  $x = t, y = t, z = t$ . Then

$$\int_C \mathbf{A} \cdot d\mathbf{r} = \int_{t=0}^1 (3t^2 - 6t^2) dt + (2t + 3t^2) dt + (1 - 4t^4) dt = 6/5$$



### Exercises:

Calculate  $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$  for the following

1-  $\mathbf{F} = \langle x^2, y^2, 0 \rangle$ ,  $C$  the semicircle from  $(2, 0)$  to  $(-2, 0)$ ,  $y \geq 0$

$$\text{Ans.} = -\frac{16}{3}$$

2-  $\mathbf{F} = \langle e^x, e^y \rangle$  clockwise along the circle with center  $(0, 0)$  from  $(1, 0)$  to  $(0, -1)$

$$\text{Ans.} = -2 \sinh 1$$

3-  $\mathbf{F} = \langle \cosh x, \sinh y, e^z \rangle$ ,  $C: \mathbf{r} = [t, t^2, t^3]$  from  $(0, 0, 0)$  to  $(\frac{1}{2}, \frac{1}{4}, \frac{1}{8})$

$$\text{Ans.} = 0.6857.$$

4-  $\mathbf{F} = \langle x, -z, 2y \rangle$  from  $(0, 0, 0)$  straight to  $(1, 1, 0)$ , then to  $(1, 1, 1)$ , back to  $(0, 0, 0)$

$$\text{Ans.} = 3/2.$$

5-  $\mathbf{F} = \langle y^2, x^2, \cos^2 z \rangle$ ,  $C: \mathbf{r} = [\cos t, \sin t, t]$  from  $(1, 0, 0)$  to  $(1, 0, 4\pi)$

$$\text{Ans.} = 2\pi.$$

### Double Integral:

*Double integrals over a region  $R$  may be evaluated by two successive integrations. We may integrate first over  $y$  and then over  $x$ . Then the formula is*

$$\iint_R f(x, y) dx dy = \int_a^b \left[ \int_{g(x)}^{h(x)} f(x, y) dy \right] dx$$

as shown by Fig.3

Here  $y = g(x)$  and  $y = h(x)$  represent the boundary curve of  $R$  (see Fig. 3 ) and, keeping  $x$  constant, we integrate  $f(x, y)$  over  $y$  from  $g(x)$  to  $h(x)$ . The result is a function of  $x$ , and we integrate it from  $x = a$  to  $x = b$  (Fig.3 ).

Similarly, for integrating first over  $x$  and then over  $y$  the formula is

$$\iint_R f(x, y) dx dy = \int_c^d \left[ \int_{p(y)}^{q(y)} f(x, y) dx \right] dy$$

as shown by Fig.4

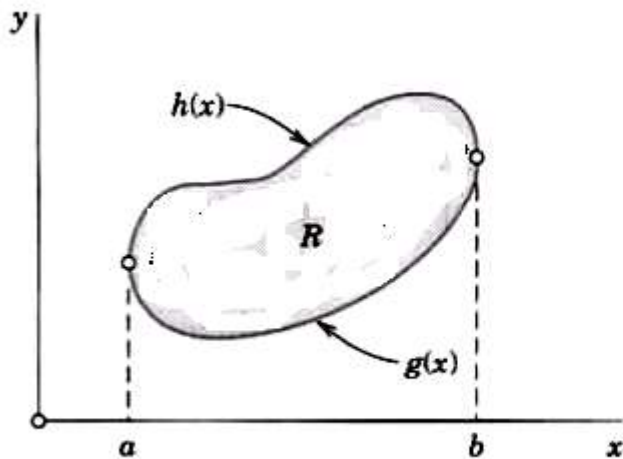


Fig. 3 Evaluation of a double integral

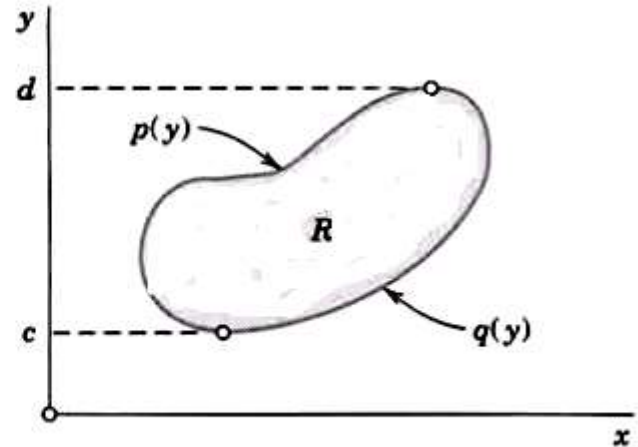


Fig. 4 Evaluation of a double integral

### Application and examples of Double integral:

Double integrals have various physical and geometric applications. For instance, the **area**  $A$  of a region  $R$  in the  $xy$ -plane is given by the double integral

$$A = \iint_R dx dy.$$

The **volume**  $V$  beneath the surface  $z = f(x, y)$  ( $> 0$ ) and above a region  $R$  in the  $xy$ -plane

$$V = \iint_R f(x, y) dx dy$$

As another application, let  $f(x, y)$  be the density (= mass per unit area) of a distribution of mass in the  $xy$ -plane. Then the **total mass**  $M$  in  $R$  is

$$M = \iint_R f(x, y) dx dy;$$

the **center of gravity** of the mass in  $R$  has the coordinates  $\bar{x}$ ,  $\bar{y}$ , where

$$\bar{x} = \frac{1}{M} \iint_R xf(x, y) dx dy \quad \text{and} \quad \bar{y} = \frac{1}{M} \iint_R yf(x, y) dx dy;$$



the **moments of inertia**  $I_x$  and  $I_y$  of the mass in  $R$  about the  $x$ - and  $y$ -axes, respectively, are

$$I_x = \iint_R y^2 f(x, y) dx dy, \quad I_y = \iint_R x^2 f(x, y) dx dy;$$

and the **polar moment of inertia**  $I_0$  about the origin of the mass in  $R$  is

$$I_0 = I_x + I_y = \iint_R (x^2 + y^2) f(x, y) dx dy.$$

### Example 5:

Calculate Volume =  $\iint_R z(x,y) dA$  for

$$z(x,y) = f(x, y) = 1 - 6x^2y \quad \text{and} \quad R: 0 \leq x \leq 2, \quad -1 \leq y \leq 1.$$

Solution:

$$\begin{aligned} \iint_R f(x, y) dA &= \int_{-1}^1 \int_0^2 (1 - 6x^2y) dx dy = \int_{-1}^1 [x - 2x^3y]_{x=0}^{x=2} dy \\ &= \int_{-1}^1 (2 - 16y) dy = [2y - 8y^2]_{-1}^1 = 4. \end{aligned}$$

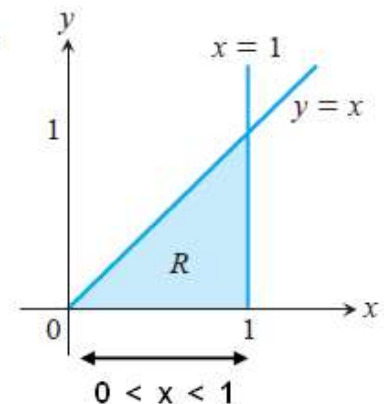
### Example 6:

Calculate  $\iint_R \frac{\sin x}{x} dA$ , where  $R$  is the triangle in the  $xy$ -plane

bounded by the  $x$ -axis, the line  $y = x$ , and the line  $x = 1$ .

Solution:

$$\begin{aligned} \int_0^1 \left( \int_0^x \frac{\sin x}{x} dy \right) dx &= \int_0^1 \left( y \frac{\sin x}{x} \right)_{y=0}^{y=x} dx = \int_0^1 \sin x dx \\ &= -\cos(1) + 1 \approx \boxed{0.46} \end{aligned}$$





### Example 7:

Let  $f(x, y) = 1$  be the mass density in the region in Fig. 5. Find the total mass, the center of gravity, and the moments of inertia  $I_x, I_y, I_0$ .

### Solution:

Total mass is

$$\begin{aligned} M &= \iint_R dx dy = \int_0^{\pi/2} \int_0^1 r dr d\theta \\ &= \int_0^{\pi/2} \frac{1}{2} d\theta = \frac{\pi}{4}. \end{aligned}$$

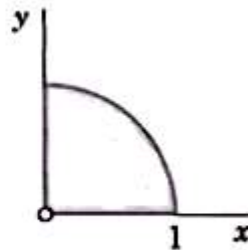


Fig. 5.

The center of gravity has the coordinates

$$\bar{x} = \frac{4}{\pi} \int_0^{\pi/2} \int_0^1 r \cos \theta r dr d\theta = \frac{4}{\pi} \int_0^{\pi/2} \frac{1}{3} \cos \theta d\theta = \frac{4}{3\pi} = 0.4244$$

In same way;

$$\bar{y} = \frac{4}{3\pi}$$

The moments of inertia are

$$\begin{aligned} I_x &= \iint_R y^2 dx dy = \int_0^{\pi/2} \int_0^1 r^2 \sin^2 \theta r dr d\theta = \int_0^{\pi/2} \frac{1}{4} \sin^2 \theta d\theta \\ &= \int_0^{\pi/2} \frac{1}{8} (1 - \cos 2\theta) d\theta = \frac{1}{8} \left( \frac{\pi}{2} - 0 \right) = \frac{\pi}{16} = 0.1963 \end{aligned}$$

Same way;

$$I_y = \frac{\pi}{16} \quad \text{and then} \quad I_0 = I_x + I_y = \frac{\pi}{8} = 0.3927.$$





## Change Cartesian integrals to polar:

Practical problems often require a change of the variables of integration in double integrals.

### For line integral:

$$\int_{x_1}^{x_2} f(x) dx = \int_{r_1}^{r_2} f(x(r)) \frac{dx}{dr} dr$$

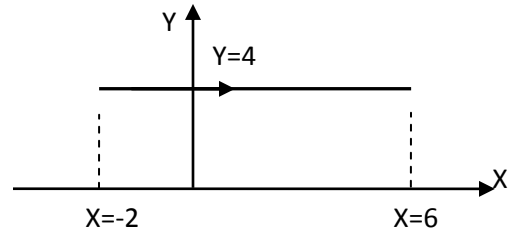
Where  $x(r)$  is continuous and has a derivative for  $r_1 \leq r \leq r_2$ .

### Example 8:

Calculate the below line integral in both Cartesian and polar forms for figure below:

In Cartesian:

$$f(x) = y = 4 \quad -2 \leq x \leq 6$$



$$\int_{x_1}^{x_2} f(x) dx = \int_{-2}^6 4 dx = 4 [x]_{-2}^6 = 32$$

In Polar:

Here  $x = r \cos \theta$  with  $\theta = \text{constant} = 0$

$$\frac{dx}{dr} = \cos \theta, \quad f(x) = 4, \quad r_1 = x_1 = -2, \quad \text{and} \quad r_2 = x_2 = 6$$

$$\text{then; } \int_{x_1}^{x_2} f(x) dx = \int_{r_1}^{r_2} f(x(r)) \frac{dx}{dr} dr = \int_{-2}^6 (4)(\cos \theta) dr$$

$$= \int_{-2}^6 (4)(\cos \theta) dr = 4 \cos 0 [r]_{-2}^6 = 32$$



### For Double – Integrals (Jacobian method):

The formula for a change of variables in double integrals from  $x, y$  to  $r, \theta$  is

$$\int \int_R f(x, y) dx dy = \int \int_{R^*} f(x(r, \theta), y(r, \theta)) \left| \frac{\partial(x, y)}{\partial(r, \theta)} \right| dr d\theta$$

Note that ;

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \frac{\partial x}{\partial r} \frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta} \frac{\partial y}{\partial r}$$

$R^*$  in the  $r\theta$ -plane such that for every  $(r, \theta)$  in  $R^*$  the corresponding point  $(x, y)$  lies in  $R$  and, conversely, to every  $(x, y)$  in  $R$  there corresponds one and only one  $(r, \theta)$  in  $R^*$ .

### Example 9:

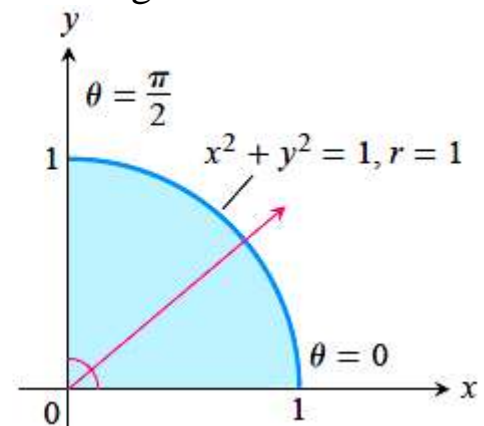
Find the Area in both Cartesian and polar form for below region:

$$A = \int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2) dy dx.$$

Integration with respect to  $y$  gives

$$\int_0^1 \left( x^2 \sqrt{1-x^2} + \frac{(1-x^2)^{3/2}}{3} \right) dx,$$

an integral difficult to evaluate without tables.



Things go better if we change the original integral to polar coordinates. Substituting  $x = r \cos \theta, y = r \sin \theta$  with  $J$ ;

$$j = (\cos \theta)(r \cos \theta) - (-r \sin \theta)(\sin \theta) = r$$

And  $r^2 = (x^2 + y^2)$  and  $0 \leq r \leq 1$  and  $0 \leq \theta \leq \pi/2$



$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2) dy dx = \int_0^{\pi/2} \int_0^1 (r^2) r dr d\theta$$

$$= \int_0^{\pi/2} \left[ \frac{r^4}{4} \right]_{r=0}^{r=1} d\theta = \int_0^{\pi/2} \frac{1}{4} d\theta = \frac{\pi}{8}.$$

**Example 10:**

**Evaluating Integrals Using Polar Coordinates**  
where  $R$  is the semicircular region bounded by the  $x$ -axis and the curve  $y = \sqrt{1-x^2}$

$$\iint_R e^{x^2+y^2} dy dx,$$

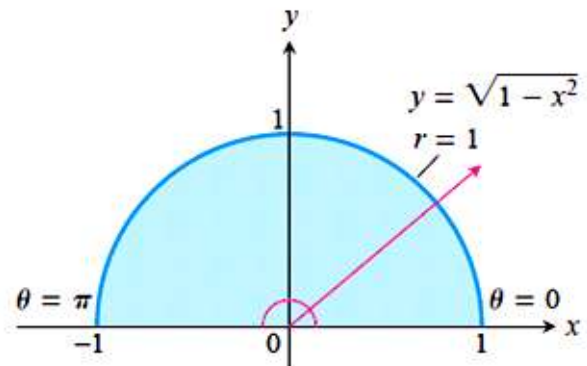
**Solution:**

using Polar coordinates . Substituting  $x = r \cos \theta$ ,  $y = r \sin \theta$  and replacing  $dy dx$  by  $r dr d\theta$  enables us to evaluate the integral as

$$\iint_R e^{x^2+y^2} dy dx = \int_0^{\pi} \int_0^1 e^{r^2} r dr d\theta$$

$$= \int_0^{\pi} \left[ \frac{1}{2} e^{r^2} \right]_0^1 d\theta = \int_0^{\pi} \frac{1}{2} (e - 1) d\theta$$

$$= \frac{\pi}{2} (e - 1).$$





## LAPLACE TRANSFORM

The Laplace transform method is a powerful method for solving linear ODEs and corresponding initial value problems, as well as systems of ODEs arising in engineering. If  $f(t)$  is a function defined for all  $t \geq 0$ , its **Laplace transform** is the integral of  $f(t)$  times  $e^{-st}$  from  $t = 0$  to  $\infty$ . It is a function of  $s$ , say,  $F(s)$ , and is denoted by  $\mathcal{L}(f)$ ; thus

$$F(s) = \mathcal{L}(f) = \int_0^{\infty} e^{-st} f(t) dt.$$

Not only is the result  $F(s)$  called the Laplace transform, but the operation just described, which yields  $F(s)$  from a given  $f(t)$ , is also called the **Laplace transform**. It is an “**integral transform**”

Furthermore, the given function  $f(t)$  is called the **inverse transform** of  $F(s)$  and is denoted by  $\mathcal{L}^{-1}(F)$ ; that is, we shall write

$$f(t) = \mathcal{L}^{-1}(F).$$

Original functions depend on  $t$  and their transforms on  $s$ - Original functions are denoted by *lowercase letters* and their transforms by the same *letters in capital*, so that  $F(s)$  denotes the transform of  $f(t)$ , and  $Y(s)$  denotes the transform of  $y(t)$ , and so on.

### Example 11:

Find the Laplace transform for the unit step function  $u(t)$ .

$$u(t) = \begin{cases} 1 & t \gg 0 \\ 0 & t < 0 \end{cases}$$

### Solution:

$$U(s) = \mathcal{L}(u(t)) = \int_0^{\infty} e^{-st} dt = -\frac{1}{s} e^{-st} \Big|_0^{\infty} = \boxed{\frac{1}{s}}$$

**Example 12:**

Let  $f(t) = e^{at}$  when  $t \geq 0$ , where  $a$  is a constant. Find  $\mathcal{L}(f)$ .

**Solution:**

$$\mathcal{L}(e^{at}) = \int_0^{\infty} e^{-st} e^{at} dt = \frac{1}{a-s} e^{-(s-a)t} \Big|_0^{\infty};$$

hence, when  $s - a > 0$ ,

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}.$$

**Linearity of the Laplace Transform**

The Laplace transform is a linear operation; that is, for any functions  $f(t)$  and  $g(t)$  whose transforms exist and any constants  $a$  and  $b$  the transform of  $af(t) + bg(t)$  exists, and

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}.$$

**Example 13:**

Find the transforms of cosh at and sinh at.

**Solution:**

Since  $\cosh at = \frac{1}{2}(e^{at} + e^{-at})$  and  $\sinh at = \frac{1}{2}(e^{at} - e^{-at})$ , we obtain from Example 2 and

$$\mathcal{L}(\cosh at) = \frac{1}{2}(\mathcal{L}(e^{at}) + \mathcal{L}(e^{-at})) = \frac{1}{2} \left( \frac{1}{s-a} + \frac{1}{s+a} \right) = \frac{s}{s^2 - a^2}$$

$$\mathcal{L}(\sinh at) = \frac{1}{2}(\mathcal{L}(e^{at}) - \mathcal{L}(e^{-at})) = \frac{1}{2} \left( \frac{1}{s-a} - \frac{1}{s+a} \right) = \frac{a}{s^2 - a^2}.$$

**Example 14:**

Find the Laplace Transform for **cos(wt)** and **sin(wt)**.

**Solution:**

$$\begin{aligned} L(\cos wt) &= \int_0^{\infty} e^{-st} \cos wt dt = \frac{1}{2} \int_0^{\infty} e^{-st} (e^{jw t} + e^{-jw t}) dt \\ &= \frac{1}{2} \int_0^{\infty} e^{-(s-jw)t} dt + \frac{1}{2} \int_0^{\infty} e^{-(s+jw)t} dt = \frac{1}{2(s-jw)} + \frac{1}{2(s+jw)} \\ &= \frac{s}{s^2 + w^2} \end{aligned}$$



Using same way;

$$L(\sin \omega t) = \frac{\omega}{s^2 + \omega^2}$$

Table (1) **Some Functions  $f(t)$  and Their Laplace Transforms  $\mathcal{L}\{f\}$**

	$f(t)$	$\mathcal{L}\{f\}$		$f(t)$	$\mathcal{L}\{f\}$
1	1	$1/s$	7	$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
2	$t$	$1/s^2$	8	$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
3	$t^2$	$2!/s^3$	9	$\cosh at$	$\frac{s}{s^2 - a^2}$
4	$t^n$ ( $n = 0, 1, \dots$ )	$\frac{n!}{s^{n+1}}$	10	$\sinh at$	$\frac{a}{s^2 - a^2}$
5	$t^a$ ( $a$ positive)	$\frac{\Gamma(a + 1)}{s^{a+1}}$	11	$e^{at} \cos \omega t$	$\frac{s - a}{(s - a)^2 + \omega^2}$
6	$e^{at}$	$\frac{1}{s - a}$	12	$e^{at} \sin \omega t$	$\frac{\omega}{(s - a)^2 + \omega^2}$

**Note:**

$\Gamma(a + 1)$  in formula 5 is the so-called *gamma function*



### First Shifting Theorem, s-Shifting

If  $f(t)$  has the transform  $F(s)$ , then  $e^{at}f(t)$  has the transform  $F(s - a)$ . In formulas,

$$\mathcal{L}\{e^{at}f(t)\} = F(s - a)$$

or, if we take the inverse on both sides,  $e^{at}f(t) = \mathcal{L}^{-1}\{F(s - a)\}$ .

Which means replacing each  $s$  by  $(s-a)$

### Examples:

Find the Laplace Transform for below functions:

1-  $f(t) = 4t e^{5t}$

Since  $L[t] = 1/s^2$  from table (1), then ;  $F(s) = \frac{4}{(s-5)^2}$

2-  $m(t) = e^{-2t} \cosh 2t$

Since  $L[\cosh 2t] = \frac{s}{s^2-4}$  then  $M(s) = \frac{s+2}{(s+2)^2-4} = \frac{s+2}{s^2+4s}$

3-  $f(t) = 6\sin 2\pi t - 3 e^{-2t} \cos 2\pi t$

Directly we may solve as  $F(s) = \frac{12\pi}{s^2+4\pi^2} - \frac{3s}{s^2+4\pi^2} = \frac{48\pi^3 - 12\pi^2 s + 12\pi s^2 - 3s^3}{s^2+4\pi^2}$

### Exercises:

Find the Laplace transforms of the following functions. Show the details of your work.

( $a, b, k, \omega, \theta$  are constants.)

1.  $t^2 - 2t$

2.  $\cos(\omega t + \theta)$

3.  $e^{3a-2bt}$

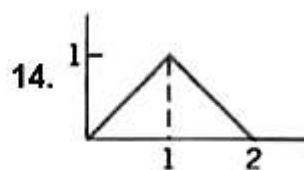
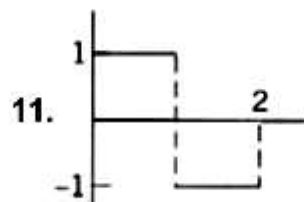
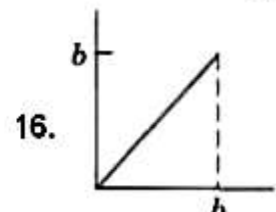
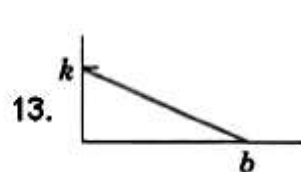
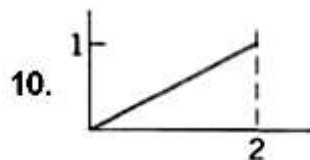
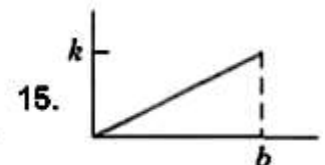
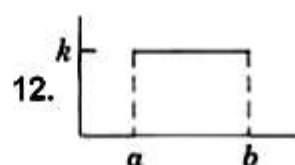
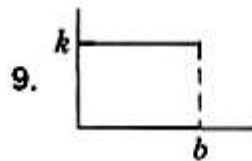
4.  $\sin^2 4t$

5.  $e^{-t} \sinh 5t$

6.  $\sin(3t - \frac{1}{2})$

7.  $\sin t \cos t$

8.  $(t + 1)^3$





### Laplace Transform of Derivatives

The transforms of the first and second derivatives of  $f(t)$  satisfy

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$$

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0).$$

Formula holds if  $f(t)$  is continuous for all  $t \geq 0$ , and  $f'(t)$  is piecewise continuous on every finite interval on the semi-axis  $t \geq 0$ .

### Laplace Transform of Integral

Let  $F(s)$  denote the transform of a function  $f(t)$  which is piecewise continuous for  $t \geq 0$  and for  $s > 0$ ,  $s > k$ , and  $t > 0$ ,

$$\mathcal{L}\left\{\int_0^t f(\tau) d\tau\right\} = \frac{1}{s}F(s) \quad \text{thus} \quad \int_0^t f(\tau) d\tau = \mathcal{L}^{-1}\left\{\frac{1}{s}F(s)\right\}$$

### Examples:

Find the Laplace Transform of the below functions:

$$1- f(t) = \frac{d}{dt} \cos 10t$$

$$F(s) = s L(\cos 10t) - \cos 0 = \frac{s^2}{s^2 + 100} - 1 = \frac{-100}{s^2 + 100}$$

$$2- f(t) = \frac{d}{dt} \sin 6t$$

$$F(s) = s L(\sin 6t) - \sin 0 = \frac{6s}{s^2 + 36} - 0 = \frac{6s}{s^2 + 36}$$

$$3- f(t) = \frac{d}{dt} 8e^{-4t}$$

$$F(s) = s L(8e^{-4t}) - 8e^0 = \frac{8s}{s + 4} - 8 = \frac{-32}{s + 4}$$

$$4- f(t) = \frac{d^2}{dt^2} 6te^{3t}$$

$$F(s) = s^2 L(6te^{3t}) - s \{6te^{3t}\}_{t=0} - \{18te^{3t} + 6e^{3t}\}_{t=0} = \frac{6s^2}{(s - 3)^2} - 6$$

$$= \frac{36s - 54}{(s - 3)^2}$$





$$5- f(t) = \int_0^t \cos 12\tau \, d\tau \quad \text{--- -- -- -- --} \rightarrow \quad F(s) = \frac{L(\cos 12t)}{s} = \frac{1}{s^2 + 144}$$

$$6- f(t) = \int_0^t 2te^{-4\tau} \, d\tau \quad \text{--- -- -- -- --} \rightarrow \quad F(s) = \frac{L(2te^{-4\tau})}{s} = \frac{2}{s(s+4)^2}$$

### Example 15:

using Laplace differential and integration properties find  $F(s)$  for ;

$$f(t) = te^{kt}$$

$$F(s) = L(te^{kt}) \quad \text{then since} \quad sF(s) - (te^{kt})_{t=0} = L\left(\frac{d}{dt} te^{kt}\right)$$

$$sF(s) = L(kte^{kt} + e^{kt}) = \frac{k}{(s-k)^2} + \frac{1}{(s-k)} = \frac{s}{(s-k)^2}$$

$$\text{therefore; } F(s) = \frac{1}{(s-k)^2}$$

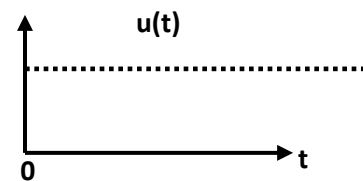
**Exercises:** Using Laplace differential and integration properties find  $F(s)$  for

**1.  $t \cos 5t$    2.  $\sin^4 t$    3.  $\sin^2 \omega t$    4.  $\cos^2 \pi t$**

**5.  $\sinh^2 at$    6.  $\cosh^2 \frac{1}{2}t$    7.  $t \sin \frac{1}{2}\pi t$**

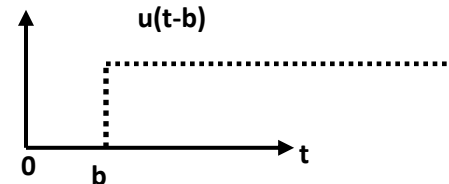
### Definition of Unit Step function:

$$u(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$



Also a unit step function with time shift is;

$$u(t-b) = \begin{cases} 1 & \text{for } t \geq b \\ 0 & \text{for } t < b \end{cases}$$



$$L[u(t)] = \int_0^{\infty} 1 \cdot e^{-st} \, dt = \frac{1}{s} \quad \text{--- -- -- -- --} \rightarrow \quad L[u(t)] = L[1] = \frac{1}{s}$$

$$L[u(t-b)] = \int_b^{\infty} 1 \cdot e^{-st} \, dt = \frac{1}{s} e^{-bs} \quad \text{--- -- -- -- --} \rightarrow \quad L[u(t-b)] = \frac{1}{s} e^{-bs}$$



**Second Shifting Theorem; Time Shifting**

If  $f(t)$  has the transform  $F(s)$ , then the “shifted function”

$$\mathcal{L}\{f(t - a)u(t - a)\} = e^{-as}F(s).$$

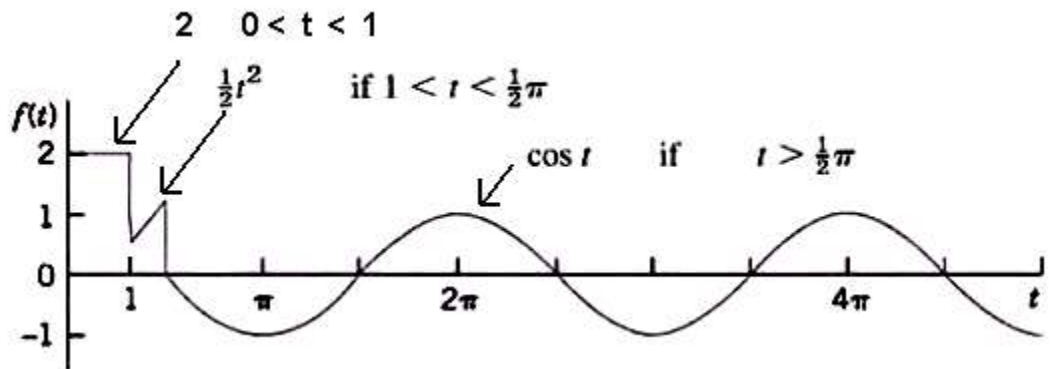
----- (1)

**Representation of functions using Unit Step:**

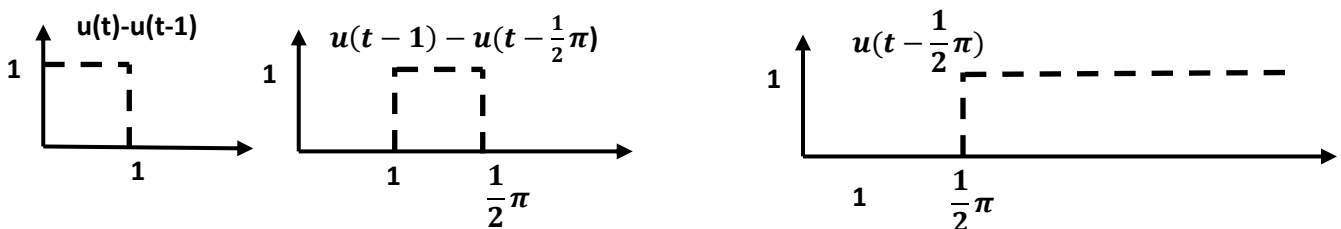
Write the following function using unit step functions and find its Laplace transform.

$$f(t) = \begin{cases} 2 & \text{if } 0 < t < 1 \\ \frac{1}{2}t^2 & \text{if } 1 < t < \frac{1}{2}\pi \\ \cos t & \text{if } t > \frac{1}{2}\pi. \end{cases}$$

The Equation is represented graphically as below



Also check the corresponding intervals using step functions:



Therefore;

$$f(t) = 2 [u(t) - u(t - 1)] + \frac{1}{2}t^2 [u(t - 1) - u(t - \frac{1}{2}\pi)] + \cos t [u(t - \frac{1}{2}\pi)]$$

To apply equation 1, we must write each term in  $f(t)$  in the form  $f(t - a)u(t - a)$ . Thus,  $2(1 - u(t - 1))$  remains as it is and gives the transform  $2(1 - e^{-s})/s$ . Then



$$\mathcal{L}\left\{\frac{1}{2}t^2u(t-1)\right\} = \mathcal{L}\left\{\frac{1}{2}(t-1)^2 + (t-1) + \frac{1}{2}\right\}u(t-1) = \left(\frac{1}{s^3} + \frac{1}{s^2} + \frac{1}{2s}\right)e^{-s}$$

$$\begin{aligned}\mathcal{L}\left\{\frac{1}{2}t^2u\left(t - \frac{1}{2}\pi\right)\right\} &= \mathcal{L}\left\{\frac{1}{2}\left(t - \frac{1}{2}\pi\right)^2 + \frac{\pi}{2}\left(t - \frac{1}{2}\pi\right) + \frac{\pi^2}{8}\right\}u\left(t - \frac{1}{2}\pi\right) \\ &= \left(\frac{1}{s^3} + \frac{\pi}{2s^2} + \frac{\pi^2}{8s}\right)e^{-\pi s/2}\end{aligned}$$

$$\mathcal{L}\left\{\cos t u\left(t - \frac{1}{2}\pi\right)\right\} = \mathcal{L}\left\{-\left(\sin\left(t - \frac{1}{2}\pi\right)\right)u\left(t - \frac{1}{2}\pi\right)\right\} = -\frac{1}{s^2 + 1}e^{-\pi s/2}.$$

Combining all parts of L-transform of  $f(t)$ ;

$$\mathcal{L}(f) = \frac{2}{s} - \frac{2}{s}e^{-s} + \left(\frac{1}{s^3} + \frac{1}{s^2} + \frac{1}{2s}\right)e^{-s} - \left(\frac{1}{s^3} + \frac{\pi}{2s^2} + \frac{\pi^2}{8s}\right)e^{-\pi s/2} - \frac{1}{s^2 + 1}e^{-\pi s/2}.$$

As another way to solve for  $F(s)$  ;

If the conversion of  $f(t)$  to  $f(t-a)$  is inconvenient, replace it by  $\mathcal{L}\{f(t)u(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)\}$

Thus,

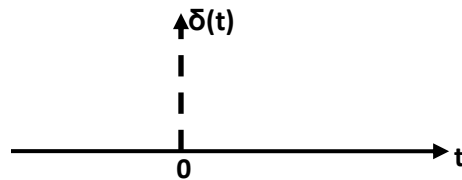
$$\mathcal{L}\left\{\frac{1}{2}t^2u(t-1)\right\} = e^{-s}\mathcal{L}\left\{\frac{1}{2}(t+1)^2\right\} = e^{-s}\mathcal{L}\left\{\frac{1}{2}t^2 + t + \frac{1}{2}\right\} = e^{-s}\left(\frac{1}{s^3} + \frac{1}{s^2} + \frac{1}{2s}\right)$$

as before. Similarly for  $\mathcal{L}\{\frac{1}{2}t^2u(t - \frac{1}{2}\pi)\}$ . Finally,

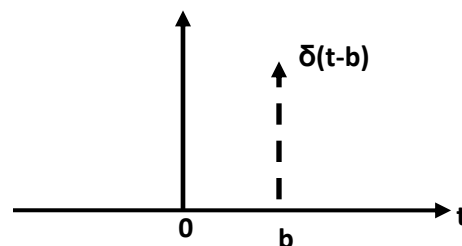
$$\mathcal{L}\left\{\cos t u\left(t - \frac{1}{2}\pi\right)\right\} = e^{-\pi s/2}\mathcal{L}\left\{\cos\left(t + \frac{1}{2}\pi\right)\right\} = e^{-\pi s/2}\mathcal{L}\{-\sin t\} = -e^{-\pi s/2}\frac{1}{s^2 + 1}. \quad \blacksquare$$

### Definition of Impulse function:

$$\delta(t) = \begin{cases} \infty & \text{for } t = 0 \\ 0 & \text{else where} \end{cases}$$



$$\delta(t-b) = \begin{cases} \infty & \text{for } t = b \\ 0 & \text{else where} \end{cases}$$





And the L-transforms are:

$$L\{\delta(t)\} = 1$$

and

$$L\{\delta(t - b)\} = e^{-bs}$$

As an example find the L-transform for the below function:

$$f(t) = 4\delta(t) - 12\delta(t-3)$$

$$F(s) = 4 - 12e^{-3s}$$

### Exercises:

Using the step function representation find the L-Transform:

- |  |  |
|--|--|
| <b>2.</b> $t$ ( $0 < t < 1$ )                    | <b>3.</b> $e^t$ ( $0 < t < 2$ )            |
| <b>4.</b> $\sin 3t$ ( $0 < t < \pi$ )            | <b>5.</b> $t^2$ ( $1 < t < 2$ )            |
| <b>6.</b> $t^2$ ( $t > 3$ )                      | <b>7.</b> $\cos \pi t$ ( $1 < t < 4$ )     |
| <b>8.</b> $1 - e^{-t}$ ( $0 < t < \pi$ )         | <b>9.</b> $t$ ( $5 < t < 10$ )             |
| <b>10.</b> $\sin \omega t$ ( $t > 6\pi/\omega$ ) | <b>11.</b> $20 \cos \pi t$ ( $3 < t < 6$ ) |
| <b>12.</b> $\sinh t$ ( $0 < t < 2$ )             | <b>13.</b> $e^{\pi t}$ ( $2 < t < 4$ )     |



## INVERSE LAPLACE TRANSFORM

### Inverse Transform using known transforms:

#### Examples:

Find the inverse L-Transform  $f(t)$  for the below functions:

$$1- F(s) = \frac{4s-3\pi}{s^2+\pi^2}$$

First to do is to simplify the function in a way so that each part may look like one of the standard Laplace-transformation in table (1)

$$\text{So } F(s) = \frac{4s}{s^2+\pi^2} - \frac{3\pi}{s^2+\pi^2} = 4 L^{-1} \left\{ \frac{s}{s^2+\pi^2} \right\} - 3 L^{-1} \left\{ \frac{\pi}{s^2+\pi^2} \right\}$$

$$\text{And then } f(t) = 4 \cos \pi t - 3 \sin \pi t$$

$$2- F(s) = \frac{10}{2s+\sqrt{2}} \quad \longrightarrow \quad F(s) = \frac{10}{2(s+\frac{\sqrt{2}}{2})} = \frac{5}{(s+\frac{\sqrt{2}}{2})}$$

$$\text{Therefore } f(t) = 5 L^{-1} \left\{ \frac{1}{(s+\frac{\sqrt{2}}{2})} \right\} = 5 e^{-\frac{\sqrt{2}}{2}t} u(t) = 5 e^{-\frac{\sqrt{2}}{2}t}$$

#### Exercises:

Given  $F(s) = \mathcal{L}(f)$ , find  $f(t)$ . Show the details. ( $L, n, k, a, b$  are constants.)

$$1. \frac{2s+16}{s^2-16}$$

$$6. \frac{1}{s^2+5} - \frac{1}{s+5}$$

$$11. \frac{7}{(s-1)^3}$$

$$2. \frac{s^4-3s^2+12}{s^5}$$

$$7. \frac{20}{(s-1)(s+4)}$$

$$12. \frac{\pi}{(s+\pi)^2}$$

$$3. \frac{n\pi L}{L^2 s^2 + n^2 \pi^2}$$

$$8. \sum_{k=1}^4 \frac{(k+1)^2}{s+k^2}$$

$$13. \frac{\sqrt{8}}{(s+\sqrt{2})^3}$$

$$4. \frac{8}{s^2+4s}$$

$$9. \frac{18s-12}{9s^2-1}$$

$$5. \frac{1}{(s-\sqrt{3})(s+\sqrt{5})}$$

$$10. \frac{1}{(s+a)(s+b)}$$



### Partial Fraction method:

This section shows how to express a rational function (a quotient of polynomials) as a sum of simpler fractions, called *partial fractions*, which are easily integrated. For instance, the rational function  $(5x - 3)/(x^2 - 2x - 3)$  can be rewritten as

$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{2}{x + 1} + \frac{3}{x - 3},$$

Success in writing a rational function  $f(x)/g(x)$  as a sum of partial fractions depends on :

- *The degree of  $f(x)$  must be less than the degree of  $g(x)$ .* That is, the fraction must be proper. If it isn't, divide  $f(x)$  by  $g(x)$  and work with the remainder term.

### Steps and cases for partial fraction method:

1. Let  $x - r$  be a linear factor of  $g(x)$ . Suppose that  $(x - r)^m$  is the highest power of  $x - r$  that divides  $g(x)$ . Then, to this factor, assign the sum of the  $m$  partial fractions:

$$\frac{f(x)}{(x - r)(x - r)^2 \cdots (x - r)^m} = \frac{A_1}{x - r} + \frac{A_2}{(x - r)^2} + \cdots + \frac{A_m}{(x - r)^m}$$

2. Let  $x^2 + px + q$  be a quadratic factor of  $g(x)$ . Suppose that  $(x^2 + px + q)^n$  is the highest power of this factor that divides  $g(x)$ . Then, to this factor, assign the sum of the  $n$  partial fractions:

$$\frac{f(x)}{(x^2 + px + q)(x^2 + px + q)^2 \cdots (x^2 + px + q)^n} = \frac{B_1x + C_1}{x^2 + px + q} + \frac{B_2x + C_2}{(x^2 + px + q)^2} + \cdots + \frac{B_nx + C_n}{(x^2 + px + q)^n}.$$

3. Set the original fraction  $f(x)/g(x)$  equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of  $x$ .
4. Equate the coefficients of corresponding powers of  $x$  and solve the resulting equations for the undetermined coefficients.

**Examples:**

Simplify the below functions using partial fraction method:

1-

$$\frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)}$$

This function can be simplified as below:

$$\frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)} = \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{C}{x + 3}$$

By which we need to find the values for A, B, and C.

$$\frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)} = \frac{A(x + 1)(x + 3) + B(x - 1)(x + 3) + C(x - 1)(x + 1)}{(x - 1)(x + 1)(x + 3)}$$

Which leads to:

$$\begin{aligned} x^2 + 4x + 1 &= A(x + 1)(x + 3) + B(x - 1)(x + 3) + C(x - 1)(x + 1) \\ &= (A + B + C)x^2 + (4A + 2B)x + (3A - 3B - C) \end{aligned}$$

The polynomials on both sides of the above equation are identical, so we equate coefficients of like powers of x obtaining

$$\text{Coefficient of } x^2: \quad A + B + C = 1$$

$$\text{Coefficient of } x^1: \quad 4A + 2B = 4$$

$$\text{Coefficient of } x^0: \quad 3A - 3B - C = 1$$

Resulting in

$$A = 3/4, B = 1/2, \text{ and } C = -1/4$$

$$\boxed{\frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)} = \frac{3}{4} \frac{1}{x - 1} + \frac{1}{2} \frac{1}{x + 1} - \frac{1}{4} \frac{1}{x + 3}}$$

2-

$$\frac{6x + 7}{(x + 2)^2}$$

Also we may re-write the function as below:

$$\frac{6x + 7}{(x + 2)^2} = \frac{A}{x + 2} + \frac{B}{(x + 2)^2} \implies 6x + 7 = A(x + 2) + B = Ax + (2A + B)$$

$$A = 6 \quad \text{and} \quad 2A + B = 12 + B = 7, \quad \text{or} \quad A = 6 \quad \text{and} \quad B = -5$$

$$\text{Therefore, } \frac{6x + 7}{(x + 2)^2} = \frac{6}{x + 2} - \frac{5}{(x + 2)^2}$$



3-

$\frac{-2x + 4}{(x^2 + 1)(x - 1)^2}$  this function can also be simplified using another form as below:

$$\frac{-2x + 4}{(x^2 + 1)(x - 1)^2} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x - 1} + \frac{D}{(x - 1)^2}$$

$$\begin{aligned} -2x + 4 &= (Ax + B)(x - 1)^2 + C(x - 1)(x^2 + 1) + D(x^2 + 1) \\ &= (A + C)x^3 + (-2A + B - C + D)x^2 + (A - 2B + C)x + (B - C + D) \end{aligned}$$

Equating coefficients of like terms gives

$$\begin{aligned} \text{Coefficients of } x^3: & 0 = A + C \\ \text{Coefficients of } x^2: & 0 = -2A + B - C + D \\ \text{Coefficients of } x^1: & -2 = A - 2B + C \\ \text{Coefficients of } x^0: & 4 = B - C + D \end{aligned}$$

We solve these equations simultaneously to find the values of  $A$ ,  $B$ ,  $C$ , and  $D$ :

$$\begin{aligned} -4 &= -2A, & \boxed{A = 2} & \text{ Subtract fourth equation from second.} \\ \boxed{C = -A = -2} & & & \text{ From the first equation} \\ \boxed{B = 1} & & & \text{ } \\ \boxed{D = 4 - B + C = 1} & & & \text{ } \end{aligned}$$

We substitute these values, obtaining

$$\frac{-2x + 4}{(x^2 + 1)(x - 1)^2} = \frac{2x + 1}{x^2 + 1} - \frac{2}{x - 1} + \frac{1}{(x - 1)^2}$$

4-

$\frac{1}{x(x^2 + 1)^2}$  proceeding with same way as below;

$$\frac{1}{x(x^2 + 1)^2} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2 + 1)^2}$$

$$\begin{aligned} 1 &= A(x^2 + 1)^2 + (Bx + C)x(x^2 + 1) + (Dx + E)x \\ &= A(x^4 + 2x^2 + 1) + B(x^4 + x^2) + C(x^3 + x) + Dx^2 + Ex \\ &= (A + B)x^4 + Cx^3 + (2A + B + D)x^2 + (C + E)x + A \end{aligned}$$





If we equate coefficients, we get the system

$$A + B = 0, \quad C = 0, \quad 2A + B + D = 0, \quad C + E = 0, \quad A = 1.$$

Solving this system gives  $A = 1$ ,  $B = -1$ ,  $C = 0$ ,  $D = -1$ , and  $E = 0$ . Thus,

$$\frac{dx}{x(x^2 + 1)^2} = \frac{1}{x} + \frac{-x}{x^2 + 1} + \frac{-x}{(x^2 + 1)^2}$$

### Inverse Laplace - Transform using partial fraction method:

Partial fraction method is used here to simplify any complicated function in S-domain to a simpler parts look like those in Table (1).

#### Examples:

Find  $f(t)$  for the below s-domain functions using partial fraction method:

$$1- F(s) = \frac{20}{(s-1)(s+4)} = \frac{A}{(s-1)} + \frac{B}{(s+4)} = \frac{A(s+4)+B(s-1)}{(s-1)(s+4)}$$

Which gives  $A=4$  and  $B=-4$  , therefore

$$F(s) = \frac{4}{(s-1)} - \frac{4}{(s+4)} \quad \text{leading to } f(t) = (4e^t - 4e^{-4t})u(t)$$

$$2- F(s) = \frac{1}{(s+a)(s+b)} = \frac{A}{(s+a)} + \frac{B}{(s+b)} = \frac{A(s+b)+B(s+a)}{(s+a)(s+b)} \quad \text{where } a \neq b$$

Which gives  $A = \frac{1}{b-a}$  and  $B = -\frac{1}{b-a}$

$$\text{Then } F(s) = \frac{\frac{1}{b-a}}{(s+a)} + \frac{-\frac{1}{b-a}}{(s+b)} \quad \text{and } f(t) = \frac{1}{b-a} (e^{-at} - \frac{1}{b-a} e^{-bt}) u(t)$$

3-

$$F(s) = \frac{20}{(s^2 + 4)(s^2 + 2s + 2)} = \frac{As + B}{s^2 + 4} + \frac{Ms + N}{s^2 + 2s + 2}$$

$$20 = (As + B)(s^2 + 2s + 2) + (Ms + N)(s^2 + 4).$$

Hence  $A = -2$ ,  $B = -2$ ,  $M = 2$ ,  $N = 6$ , and then;

$$F(s) = \frac{-2s - 2}{s^2 + 4} + \frac{2s + 6}{s^2 + 2s + 2} = \frac{-2s}{s^2 + 4} - \frac{2}{s^2 + 4} + \frac{2s + 6}{s^2 + 2s + 1 + 1}$$

$$F(s) = \frac{-2s}{s^2 + 4} - \frac{2}{s^2 + 4} + \frac{2s + 2 + 4}{(s + 1)^2 + 1}$$



$$F(s) = \frac{-2s}{s^2 + 4} - \frac{2}{s^2 + 4} + \frac{2(s + 1)}{(s + 1)^2 + 1} + \frac{4}{(s + 1)^2 + 1}$$

Which gives;

$$f(t) = \{-2 \cos 2t - \sin 2t + 2 e^{-t} \cos t + 4 e^{-t} \sin t\} u(t)$$

### Exercises:

Using partial fraction find the inverse L-transform for:

1.  $\frac{15}{s^2 + 4s + 29}$

5.  $\frac{1}{s^2 + s/2}$

9.  $\frac{10}{s^3 - \pi s^2}$

2.  $\frac{4s - 2}{s^2 - 6s + 18}$

6.  $\frac{1}{s^3 - ks^2}$

10.  $\frac{1}{s^4 + s^2}$

7.  $\frac{5}{s^3 - 5s}$

11.  $\frac{2}{s^3 + 9s}$

3.  $\frac{\pi}{s^2 + 10\pi s + 24\pi^2}$

8.  $\frac{1}{s^4 - 4s^2}$

12.  $\frac{1}{s^4 + \pi^2 s^2}$

4.  $\frac{2s - 56}{s^2 - 4s - 12}$



## ORDINARY DIFFERENTIAL EQUATIONS SOLUTION

### ( ODEs )

We begin with the simplest ODEs, called ODEs *of the first order* because they involve only the first derivative of the unknown function, no higher derivatives. Our usual notation for the unknown function will be  $y(x)$ , or  $y(t)$  if the independent variable is time  $t$ .

An **ordinary differential equation (ODE)** is an equation that contains one or several derivatives of an unknown function, which we usually call  $y(x)$  (or sometimes  $y(t)$  if the independent variable is time  $t$ ). The equation may also contain  $y$  itself, known functions of  $x$  (or  $t$ ), and constants. For example,

$$y' = \cos x,$$

$$y'' + 9y = 0,$$

$$x^2 y''' y' + 2e^x y'' = (x^2 + 2)y^2$$

are ordinary differential equations (ODEs).

we shall consider **first-order ODEs**. Such equations contain only the first derivative  $y'$  and may contain  $y$  and any given functions of  $x$ . Hence we can write them as

$$F(x, y, y') = 0$$

or often in the form

$$y' = f(x, y).$$

### **Example16:**

Consider the below ODE :

$$\dot{y} = 3y \quad \text{where } y=f(t) \text{ and } \dot{y} = \frac{dy}{dt} \text{ with initial value } y(0)=5.7$$

Now by taking L-transform for both sides of  $\dot{y} = 3y$ , get;

$$sY(s) - y(0) = 3Y(s) \text{ and then } Y(s) = \frac{y(0)}{s-3} = \frac{5.7}{s-3}$$

Using inverse L-transform  $y(t) = 5.7 e^{3t} u(t)$



Or simply  $y(t) = 5.7 e^{3t}$

**Example17:**

Solve the following ODEs using inverse L-transform.

1-  $\dot{y} = 0.5y$  with condition  $y(2) = 2$

Solution: Taking L-transform for both sides;

$$sY(s) - y(0) = 0.5Y(s) \quad \text{then } Y(s) = \frac{y(0)}{s - 0.5} = k \frac{1}{s - 0.5}$$

Now;  $y(t) = k e^{0.5t}$  to find the value of k, use  $y(2)=2$ , get

$$y(2) = k e^1 \quad \rightarrow \quad k = \frac{2}{e} = 0.736$$

Therefore;  $y(t) = 0.736 e^{0.5t}$

2-  $\dot{y} = y - t$  with condition  $y(0) = 3$

$$sY(s) - y(0) = Y(s) - \frac{1}{s^2} \quad \text{then } Y(s) = \frac{y(0) - \frac{1}{s^2}}{s - 1} = \frac{3}{s - 1} - \frac{1}{s^2(s - 1)}$$

Now using partial fraction  $\frac{1}{s^2(s-1)} = \frac{As+B}{s^2} + \frac{C}{(s-1)}$

Which gives  $A=B=-1$  and  $C=1$

Therefore  $\frac{1}{s^2(s-1)} = \frac{-s-1}{s^2} + \frac{1}{(s-1)} = -\frac{1}{s} - \frac{1}{s^2} + \frac{1}{(s-1)}$  and then substituting in  $Y(s)$

$$Y(s) = \frac{3}{s - 1} + \frac{1}{s} + \frac{1}{s^2} - \frac{1}{(s - 1)}$$

Taking inverse L-T;

$$y(t) = 3e^t + 1 + t - e^t = 2e^t + 1 + t$$

**Exercise:**

Solve the following ODEs using inverse L-transform.

1-  $\dot{y} = y \tan t$  with condition  $y(0) = \frac{1}{2}\pi$

2-  $\dot{y} + 2ty = 0$  with condition  $y(1) = \frac{1}{e}$



## Separable ODEs:

Many practically useful ODEs can be reduced to the form  $g(y)y' = f(x)$  by purely algebraic manipulations. Then we can integrate on both sides with respect to  $x$ , obtaining

$$\int g(y) y' dx = \int f(x) dx + c.$$

On the left we can switch to  $y$  as the variable of integration. By calculus,  $y' dx = dy$ , so that

$$\int g(y) dy = \int f(x) dx + c.$$

If  $f$  and  $g$  are continuous functions, the integrals in (3) exist, and by evaluating them we obtain a general solution of  $g(y)y' = f(x)$ . This method of solving ODEs is called the **method of separating variables**, and  $g(y)y' = f(x)$  is called a **separable equation**, because in

$$\int g(y) dy = \int f(x) dx + c.$$

the variables are now separated:  $x$  appears only on the right and  $y$  only on the left.

### Example 18:

Find the solution for below ODE:

$$\dot{y} = 1 + y^2$$

It is desired to put each variables together in one side as below;

$$\frac{dy}{dx} = 1 + y^2 \rightarrow \frac{dy}{(1 + y^2)} = dx$$

Now integrating both sides give;

$$\int \frac{1}{(1 + y^2)} dy = \int dx \rightarrow \tan^{-1}y = x + c \rightarrow y = \tan(x + c)$$

$C$  is a constant of integration.



## **Second Order Differential Equation:**

Linear ODEs *of the second order* are the most important ones because of their applications in mechanical and electrical engineering .

## **Homogenous Linear ODEs of Second Order:**

A second-order ODE is called **linear** if it can be written

$$y'' + p(x)y' + q(x)y = r(x)$$

and **nonlinear** if it cannot be written in this form.

If  $r(x) \equiv 0$  (that is,  $r(x) = 0$  for all  $x$  considered; read “ $r(x)$  is identically zero”), then

$$y'' + p(x)y' + q(x)y = 0$$

and is called **homogeneous**. If  $r(x) \neq 0$ , then (1) is called **nonhomogeneous**.

For example; a homogeneous linear ODE is  $xy'' + y' + xy = 0$ ,

in standard form

$$y'' + \frac{1}{x}y' + y = 0.$$

Which could be written in general as :

$$P(x)y'' + Q(x)y' + R(x)y = 0.$$

We also assume that  $P(x)$  is never zero for any  $x$

**The Superposition Principle** : If  $y_1(x)$  and  $y_2(x)$  are two solutions to the linear homogeneous equation , then for any constants  $c_1$  and  $c_2$ , the function

$$y(x) = c_1y_1(x) + c_2y_2(x)$$

is also a solution to Equation  $P(x)y'' + Q(x)y' + R(x)y = 0$ .



### Constant-coefficient homogenous Equations:

Suppose we wish to solve the second-order homogeneous differential equation

$$\boxed{ay'' + by' + cy = 0} \quad \text{where } a, b, \text{ and } c \text{ are constants.}$$

With below auxiliary equation (characteristics equation) :

$$\boxed{ar^2 + br + c = 0.}$$

Which have roots ;

$$\boxed{r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}} \quad \text{and} \quad \boxed{r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}}$$

There are three cases to consider which depend on the value of the discriminant  $\underline{\underline{b^2 - 4ac}}$ .

#### ● Case 1: $b^2 - 4ac > 0$

If  $r_1$  and  $r_2$  are two real and unequal roots to the auxiliary equation  $ar^2 + br + c = 0$ , then

$$\boxed{y = c_1e^{r_1x} + c_2e^{r_2x}}$$

is the general solution to

$$ay'' + by' + cy = 0.$$

#### ● Case 2: $b^2 - 4ac = 0$

In this case  $r_1 = r_2 = r = -b/2a$ . (Equal Roots)

If  $r$  is the only (repeated) real root to the auxiliary equation

$$ar^2 + br + c = 0, \quad \text{then} \quad \boxed{y = c_1e^{rx} + c_2xe^{rx}}$$

is the general solution to  $ay'' + by' + cy = 0$ .



### Case 3: $b^2 - 4ac < 0$

In this case the auxiliary equation has two complex roots

$r_1 = \alpha + i\beta$  and  $r_2 = \alpha - i\beta$ , where  $\alpha$  and  $\beta$  are real numbers (These real numbers are  $\alpha = -b/2a$  and  $\beta = \sqrt{4ac - b^2}/2a$ .)

Now; If  $r_1 = \alpha + i\beta$  and  $r_2 = \alpha - i\beta$  are two complex roots to the auxiliary equation  $ar^2 + br + c = 0$ , then

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$$

is the general solution to  $ay'' + by' + cy = 0$ .

### Examples:

1-

Find the general solution of the differential equation  $y'' - y' - 6y = 0$ .

with initial conditions of  $y(0)=3$  and  $y'(0)=-1$ .

Solution: Substitution of  $y = e^{rx}$  into the differential equation yields the auxiliary equation  $r^2 - r - 6 = 0$ ,

which factors as  $(r - 3)(r + 2) = 0$ .

The roots are  $r_1 = 3$  and  $r_2 = -2$ . Thus, the general solution is  $y = c_1 e^{3x} + c_2 e^{-2x}$ .

Now using the initial conditions given;

$y(0)=3 = c_1 + c_2$  and  $y'(0)=-1 = 3c_1 - 2c_2$  and by combining the two equations we have:  $c_1=1$  and  $c_2=2$ . Therefore ;

$$y = e^{3x} + 2e^{-2x}$$

2-

Find the general solution to  $y'' + 4y' + 4y = 0$ . with  $y(0)=-2$  and  $y(-0.5)=e^1$

Solution:

The auxiliary equation is  $r^2 + 4r + 4 = 0$ , which factors into

$$(r + 2)^2 = 0.$$

Thus,  $r = -2$  is a double root. Therefore, the general solution is  $y = c_1 e^{-2x} + c_2 x e^{-2x}$ .

Also if we substituted the initial values we obtain :

$$y = -2e^{-2x} - 6x e^{-2x}$$





3- Find the general solution to the differential equation  $y'' - 4y' + 5y = 0$ , with initial conditions  $y(0)=2$  and  $y\left(\frac{\pi}{4}\right) = 2\sqrt{2} e^{\sqrt{2}}$

Solution: The auxiliary equation is  $r^2 - 4r + 5 = 0$ .

The roots are the complex pair  $r = (4 \pm \sqrt{16 - 20})/2$

or  $r_1 = 2 + i$

and  $r_2 = 2 - i$ . Thus,  $\alpha = 2$  and  $\beta = 1$

give the general solution  $y = e^{2x}(c_1 \cos x + c_2 \sin x)$ .

Using the given initials;

$$y = e^{2x} (2\cos x + 2\sin x)$$

### H.Ws:

1- Find the particular solution to the initial value problem

$$y'' - 2y' + y = 0, \quad y(0) = 1, \quad y'(0) = -1.$$

Ans.  $y = e^x - 2xe^x$ .

2- Solve the boundary value problem  $y'' + 4y = 0$ ,  $y(0) = 0$ ,  $y\left(\frac{\pi}{12}\right) = 1$ .

Ans.  $y = 2 \sin 2x$ .

### Exercises:

In Exercises 1–30, find the general solution of the given equation.

1.  $y'' - y' - 12y = 0$

2.  $3y'' - y' = 0$

3.  $y'' + 3y' - 4y = 0$

4.  $y'' - 9y = 0$

5.  $y'' - 4y = 0$

6.  $y'' - 64y = 0$

7.  $2y'' - y' - 3y = 0$

8.  $9y'' - y = 0$

9.  $8y'' - 10y' - 3y = 0$

10.  $3y'' - 20y' + 12y = 0$

11.  $y'' + 9y = 0$

12.  $y'' + 4y' + 5y = 0$

13.  $y'' + 25y = 0$

14.  $y'' + y = 0$

15.  $y'' - 2y' + 5y = 0$

16.  $y'' + 16y = 0$



17.  $y'' + 2y' + 4y = 0$

19.  $y'' + 4y' + 9y = 0$

21.  $y'' = 0$

23.  $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$

25.  $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$

27.  $4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = 0$

29.  $9\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + y = 0$

18.  $y'' - 2y' + 3y = 0$

20.  $4y'' - 4y' + 13y = 0$

22.  $y'' + 8y' + 16y = 0$

24.  $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$

26.  $4\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 9y = 0$

28.  $4\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 0$

30.  $9\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 4y = 0$

In Exercises 31–40, find the unique solution of the second-order initial value problem.

31.  $y'' + 6y' + 5y = 0, \quad y(0) = 0, \quad y'(0) = 3$

32.  $y'' + 16y = 0, \quad y(0) = 2, \quad y'(0) = -2$

33.  $y'' + 12y = 0, \quad y(0) = 0, \quad y'(0) = 1$

34.  $12y'' + 5y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = -1$

35.  $y'' + 8y = 0, \quad y(0) = -1, \quad y'(0) = 2$

36.  $y'' + 4y' + 4y = 0, \quad y(0) = 0, \quad y'(0) = 1$

37.  $y'' - 4y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 0$

38.  $4y'' - 4y' + y = 0, \quad y(0) = 4, \quad y'(0) = 4$

39.  $4\frac{d^2y}{dx^2} + 12\frac{dy}{dx} + 9y = 0, \quad y(0) = 2, \quad \frac{dy}{dx}(0) = 1$

40.  $9\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 4y = 0, \quad y(0) = -1, \quad \frac{dy}{dx}(0) = 1$



In Exercises 41–55, find the general solution.

41.  $y'' - 2y' - 3y = 0$

43.  $4y'' + 4y' + y = 0$

45.  $4y'' + 20y = 0$

47.  $25y'' + 10y' + y = 0$

49.  $4y'' + 4y' + 5y = 0$

51.  $16y'' - 24y' + 9y = 0$

53.  $9y'' + 24y' + 16y = 0$

55.  $6y'' - 5y' - 4y = 0$

42.  $6y'' - y' - y = 0$

44.  $9y'' + 12y' + 4y = 0$

46.  $y'' + 2y' + 2y = 0$

48.  $6y'' + 13y' - 5y = 0$

50.  $y'' + 4y' + 6y = 0$

52.  $6y'' - 5y' - 6y = 0$

54.  $4y'' + 16y' + 52y = 0$

In Exercises 56–60, solve the initial value problem.

56.  $y'' - 2y' + 2y = 0, \quad y(0) = 0, \quad y'(0) = 2$

57.  $y'' + 2y' + y = 0, \quad y(0) = 1, \quad y'(0) = 1$

58.  $4y'' - 4y' + y = 0, \quad y(0) = -1, \quad y'(0) = 2$

59.  $3y'' + y' - 14y = 0, \quad y(0) = 2, \quad y'(0) = -1$

60.  $4y'' + 4y' + 5y = 0, \quad y(\pi) = 1, \quad y'(\pi) = 0$



## **None- Homogenous Linear ODEs of Second Order:**

Suppose we wish to solve the nonhomogeneous equation

$$ay'' + by' + cy = G(x),$$

where  $a$ ,  $b$ , and  $c$  are constants and  $G$  is continuous over some open interval  $I$ . Let  $y_c = c_1y_1 + c_2y_2$  be the general solution to the associated **complementary equation**

$$ay'' + by' + cy = 0.$$

The general solution  $y = y(x)$  to the nonhomogeneous differential equation (1) has the form  $y = y_c + y_p$ ,

where the **complementary solution**  $y_c$  is the general solution to the associated homogeneous equation  $ay'' + by' + cy = 0$ .

and  $y_p$  is any **particular solution** to the nonhomogeneous equation

$$ay'' + by' + cy = G(x)$$

### **Examples:**

1-

Solve the nonhomogeneous equation  $y'' - 2y' - 3y = 1 - x^2$ .

**Solution:** The auxiliary equation for the complementary equation  $y'' - 2y' - 3y = 0$  is

$$r^2 - 2r - 3 = (r + 1)(r - 3) = 0.$$

It has the roots  $r = -1$  and  $r = 3$  giving the complementary solution

$$y_c = c_1e^{-x} + c_2e^{3x}.$$

Now  $G(x) = 1 - x^2$  is a polynomial of degree 2. It would be reasonable to assume that a particular solution to the given nonhomogeneous equation is also a polynomial of degree 2 because if  $y$  is a polynomial of degree 2, then  $y'' - 2y' - 3y$  is also a polynomial of degree 2. So we seek a particular solution of the form

$$y_p = Ax^2 + Bx + C.$$

We need to determine the unknown coefficients  $A$ ,  $B$ , and  $C$ . When we substitute the polynomial  $y_p$  and its derivatives into the given nonhomogeneous equation, we obtain

$$2A - 2(2Ax + B) - 3(Ax^2 + Bx + C) = 1 - x^2$$



or, collecting terms with like powers of  $x$ ,

$$-3Ax^2 + (-4A - 3B)x + (2A - 2B - 3C) = 1 - x^2.$$

This last equation holds for all values of  $x$  if its two sides are identical polynomials of degree 2. Thus, we equate corresponding powers of  $x$  to get

$$-3A = -1, \quad -4A - 3B = 0, \quad \text{and} \quad 2A - 2B - 3C = 1.$$

These equations imply in turn that  $A = 1/3$ ,  $B = -4/9$ , and  $C = 5/27$ . Substituting these values into the quadratic expression for our particular solution gives

$$y_p = \frac{1}{3}x^2 - \frac{4}{9}x + \frac{5}{27}.$$

the general solution to the nonhomogeneous equation is

$$y = y_c + y_p = c_1e^{-x} + c_2e^{3x} + \frac{1}{3}x^2 - \frac{4}{9}x + \frac{5}{27} \quad \blacksquare$$

Find a particular solution of  $y'' - y' = 2 \sin x$ .

**Solution:** If we try to find a particular solution of the form  $y_p = A \sin x$  and substitute the derivatives of  $y_p$  in the given equation, we find that  $A$  must satisfy the equation

$$-A \sin x + A \cos x = 2 \sin x$$

for all values of  $x$ . Since this requires  $A$  to equal both  $-2$  and  $0$  at the same time, we conclude that the nonhomogeneous differential equation has no solution of the form  $A \sin x$ .

It turns out that the required form is the sum  $y_p = A \sin x + B \cos x$ .

The result of substituting the derivatives of this new trial solution into the differential equation is

$$-A \sin x - B \cos x - (A \cos x - B \sin x) = 2 \sin x$$

or

$$(B - A) \sin x - (A + B) \cos x = 2 \sin x.$$

This last equation must be an identity. Equating the coefficients for like terms on each side then gives

$$B - A = 2 \quad \text{and} \quad A + B = 0.$$

Simultaneous solution of these two equations gives  $A = -1$  and  $B = 1$ . Our particular solution is

$$y_p = \cos x - \sin x. \quad \blacksquare$$



### Solving ODEs using Laplace Transform:

The Laplace transform method may also be used for solving systems of ODEs, as we shall explain in terms of typical applications. We consider a first-order linear system with constant coefficients

#### Example 19:

Find the solution for  $y_1$  and  $y_2$ ;

$$y_1' = -\frac{8}{100}y_1 + \frac{2}{100}y_2 + 6, \quad y_2' = \frac{8}{100}y_1 - \frac{8}{100}y_2.$$

The initial conditions are  $y_1(0) = 0, y_2(0) = 150$ .

Taking Laplace transform for each side of each equation, get;

$$\begin{aligned} (-0.08 - s)Y_1(s) + 0.02Y_2(s) &= -\frac{6}{s} \\ 0.08Y_1(s) + (-0.08 - s)Y_2(s) &= -150. \end{aligned}$$

Solving both for  $Y_1(s)$  and  $Y_2(s)$ ;

$$Y_1 = \frac{9s + 0.48}{s(s + 0.12)(s + 0.04)} = \frac{100}{s} - \frac{62.5}{s + 0.12} - \frac{37.5}{s + 0.04}$$

$$Y_2 = \frac{150s^2 + 12s + 0.48}{s(s + 0.12)(s + 0.04)} = \frac{100}{s} + \frac{125}{s + 0.12} - \frac{75}{s + 0.04}.$$

Partial Fraction

By Inverse L-Transform ;

$$y_1 = 100 - 62.5e^{-0.12t} - 37.5e^{-0.04t}$$

$$y_2 = 100 + 125e^{-0.12t} - 75e^{-0.04t}.$$

#### Exercises:

Using the Laplace transform and showing the details of your work, solve the initial value problem:

$$1. \quad \begin{aligned} y_1' &= -y_1 - y_2, & y_2' &= y_1 - y_2, \\ y_1(0) &= 0, & y_2(0) &= 1 \end{aligned}$$

$$2. \quad \begin{aligned} y_1' &= 5y_1 + y_2, & y_2' &= y_1 + 5y_2, \\ y_1(0) &= 1, & y_2(0) &= -3 \end{aligned}$$

